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Generalized Möbius Inversion with Applications to Integral Equations and Interpolation Process*

In Memory of the late Professor Loo-Keng Hua (华罗庚)

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Abstract. The object of this expository paper is to establish a kind of generalized Möbius-Rota inversion formula by the aid of non-standard analysis. Certain constructive applications to Volterra type integral equations and to equidistant smooth interpolation process are expounded in some detail. Moreover, some of our earlier results (cf. [1], [2], [4]) are refined and improved.

§ 1 Introduction

Throughout this paper (except § 6) the method of non-standard analysis will be utilized. As usual, R and *R will denote, respectively, the ordinary real number field and the non-standard real field. It is always assumed that R has been embedded in *R. Moreover, N and *N denote the sets of integers contained in R and *R respectively, so that $^*N \setminus N$ is the set of non-standard integers. We will make frequent use of the fundamental integer $\omega \in {}^*N \setminus N$. Accordingly $\varepsilon = 1/\omega$ is a positive infinitesimal (cf. [5], [6], [10]).

In accordance with non-standard analysis, every ordinary function $f: R \to R$ has a natural extension $^*f: {}^*R \to {}^*R$ such that

$$\mathrm{st}({}^*f({}^*x)) = f(x) \quad \text{with } \mathrm{st}({}^*x) = {}^\circ({}^*x) = x \in R,$$

where st(·) or $^\circ$(·) means taking standard part of (·). As a convention we will always retain the original notation f in place of $^*f$ whenever its domain of definition has been extended apparently to *R. Similarly, the denotation $(a, b) \in {}^*R$ just implies that the ordinary interval (a, b) contained in R has been extended to $^*(a, b) \in {}^*R$.

What we want to formulate is the basic concept of an ε-partitioned structure defined on some infinite poset which is not locally finite. Our main idea is that the ε-partitioned structure may be viewed as a kind of semi-discrete structure, so that general Möbius function pairs as well as general inversion formulae can be established within a kind of incidence algebra defined on the struc-

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ture. These will be expounded in the next three sections.

Taking this occasion, I would like to mention that I learned the classical Möbius inversion theorem from a course on number theory given by the late Professor L. K. Hua during 1943—44. Especially my earlier interest was much stimulated by Hua's impressive talk about his discovery of the use by Japanese of Möbius inversion in the military coding during the Sino-Japanese war time.

§ 2 Some Basic Concepts

We will need the following

Definition 1. Let $L_1, \cdots, L_s$ be distinct lines embedded in the r-dimensional nonstandard Euclidean space $^*R^r$ ($r \leq s$). These straight or curved lines may intersect at a finite number of points. Given a poset $S \equiv (S, \leq)$. Suppose that all the elements of S can be represented by the points of some sets or intervals on L_i's, where the order relation $\leq$ is defined in the usual way, namely in accordance with the natural ordering along each line L_i ($1 \leq i \leq s$). Then S is called a point representable poset with chains defined on lines $\{L_i\}$.

In what follows we will always assume that every poset $(S, \leq)$ to be concerned has been embedded in $^*R^r$. Now suppose that I is an open or closed interval of $^*R$ that forms chain in S. If I is divided into an infinitude of ε-intervals, so called the fine partition with ε-length, then I is said to have an ε partitioned structure $\hat{I} \equiv \hat{I}_\varepsilon$. More precisely we have the following

Definitton 2. If all all the maximal chains of a point-representable poset $S \equiv (S, \leq)$ are intervals or unions of intervals that have been divided into fine partitions with ε-length, then all the division points (end points of ε-intervals) are said to form an ε partitioned structure $\hat{S} \equiv \hat{S}_\varepsilon$.

Evidently $\hat{S} \equiv (\hat{S}, \leq)$ is a subposet of S, in which the order relation is just the same as that of S.

By $O(\omega)$ we denote generally a positive integer in $^*N \setminus N$ such that $O(\omega) \leq M\omega$, M being a finite positive number.

Definition 3. A point-representable poset $S \equiv (S, \leq)$ is said to possess property (ω), if the following three conditions are satisfied:

(i) All the maximal chains of S are intervals or unions of intervals.

(ii) Every maximal chain of $\hat{S} \equiv \hat{S}_\varepsilon$ has a minimal element.

(iii) Every maximal chain of $\hat{S}$ has a cardinality $O(\omega)$.

Example 1. A finite interval $I \equiv (a, b) \in {}^*R$ has its ε partitioned structure $\hat{I}$ given by

$$\hat{I} := \{a + j\varepsilon \mid j \in {}^*N,\ j \leq (b-a)/\varepsilon\}.$$

The cardinality is given by the integer part card $\hat{I} := [(b-a)\omega] \in {}^*N \setminus N$. Clearly $[(b-a)\omega]=O(\omega)$. Moreover, $[a+j\varepsilon, a+n\varepsilon]$ with $j<n$ may be called a chain of $\hat{I}$ with total length $(n-j)$.

Given $x, y\in\hat{S}$ with $x<y$. By a segment $\langle x, y\rangle$ we mean a set of all elements $z\in\hat{S}$ such that $x\leq z\leq y$. Since we are always concerned with point-representable poset $(S, \leq)$ having property (ω), we see that every segment of $\hat{S}$ can contain at most $O(\omega)$ elements (points). Most frequently we will deal with summations of the types

$$(\mathrm{I})\quad \sum_{x\leq z\leq y}(\cdot),\qquad (\mathrm{II})\quad \sum_{z\leq y}(\cdot),\qquad (\mathrm{III})\quad \sum_{k=0}^{m}(\cdot),\quad (m=O(\omega)).$$

Here the summation condition $x\leq z\leq y$ means that z ranges over all the elements of the segment $\langle x, y\rangle$ in $\hat{S}$. Similarly, $z\leq y$ indicates that z ranges over all the chains with y as an upper bound. Clearly all these summations consist of only $O(\omega)$ terms, and they may be called $O(\omega)$-summations. More generally, repeated $O(\omega)$-summations are also called $O(\omega)$-summations.

§ 3 An Incidence Algebra defined on $(\hat{S}\times\hat{S}\to{}^{*}R)$

We are now going to define a kind of incidence algebra on $(\hat{S}\times\hat{S}\to{}^{*}R)$. First, as in the ordinary case, one may define delta function δ, incidence function λ and zeta function ζ on $\hat{S}\times\hat{S}$ respectively, viz.

$$\delta(x, y)=\begin{cases}1, & (x=y)\\ 0, & (x\neq y),\end{cases}\quad \lambda(x, y)=\begin{cases}1, & (x<y)\\ 0, & (x\nless y),\end{cases}\quad \zeta(x, y)=\begin{cases}1, & (x\leq y)\\ 0, & (x\nleq y),\end{cases}$$

where $(x, y)\in\hat{S}\times\hat{S}$, and $\zeta=\delta+\lambda$.

All the functions $f(x, y)\in(\hat{S}\times\hat{S}\to{}^{*}R)$ with the property that $f(x, y)=0$ for $x\nleq y$ form an aggregate $\mathfrak{C}$. For any elements f and g of $\mathfrak{C}$, the addition $f+g$ as well as multiplication by a scalar in ${}^{*}R$ may be defined in usual way. However, in order to define a kind of generalized Dedekind product on $\mathfrak{C}$, certain points regarding formal operations with $O(\omega)$-summations should be made clear.

For dealing with formal operations and non-standard analytic computations concerning $O(\omega)$-summations, we have to make use of a fundamental postulate, namely

Formal equality postulate: Two analytic expressions composed of $O(\omega)$-summations are formally equal, namely they are equivalent in ${}^{*}R$, if there can be found a complete one-to-one correspondence between the terms contained in both expressions.

In applying this postulate to derive equalities, one has to find that a general term of the same form should occur exactly once in either of the two expressions, and that the ranges of summation variables contained in both expressions are exactly the same one.

We have to distinguish two kinds of formal computability.

Definition 4 Any given function $f\in(\hat{S}\times\hat{S}\to{}^*R)$ is said to be formally computable in $^*R$ with respect to $O(\omega)$-summations, if for every given $y\in\hat{S}$ and $(t, y)\in\hat{S}\times\hat{S}$ the $O(\omega)$-summations of types (Ⅰ) and (Ⅱ) with $f(x, y)$ as summands have definite meanings and can be formally evaluated in $^*R$, so that

$$\sum_{t\le x\le y} f(x, y)\in{}^*R, \qquad \sum_{x\le y} f(x, y)\in{}^*R.$$

In particular, if

$$\Big(\sum_{t\le x\le y} f(x, y)\Big)\in R, \qquad \Big(\sum_{x\le y} f(x, y)\Big)\in R,$$

then $f(x, y)$ is said to be finitely computable in $^*R$.

Certainly one can also give a definition for the formal computability of $a_k\equiv a(k)\in({}^*N\to{}^*R)$ with respect to the $O(\omega)$-summation of type (Ⅲ) in $^*R$. Still we will require a concept for formal computability in the general sense.

Definition 5. If two or more functions of $(\hat{S}\times\hat{S}\to{}^*R)$ or of $(\hat{S}\to{}^*R)$ appear in the summand of an $O(\omega)$-summation of either type (Ⅰ) or type (Ⅱ), and if the sum can be evaluated in $^*R$ using non-standard analysis plus the formal equality postulate, then these functions are said to be formally computable in the general sense. In particular, any given functions $f_i\in(\hat{S}\times\hat{S}\to{}^*R)$ $(i=1,2,3,\cdots)$ are called formal computable elements in the general sense, if $f_1(x, z)f_2(z, y)$, $f_1(x, z)f_2(z, y)f_3(y, u)$, etc. are all formally computable with respect to $O(\omega)$-summations in $^*R$.

Dedekind product and associative law. Let $f, g\in\mathfrak{F}$ be formally computable in the general sense. Then a kind of Dedekind product $p=f\cdot g$ may be defined formally by $p(x, y)=(f\cdot g)(x, y)$ for every segment $\langle x, y\rangle$ of $\hat{S}$:

$$(f\cdot g)(x, y):=\sum_{x\le z\le y} f(x, z)g(z, y).$$

Suppose that f, g, h are all formally computable. Then the associative law $(f\cdot g)\cdot h=f\cdot(g\cdot h)$ holds. In fact, for any given $(t, y)\in\hat{S}\times\hat{S}$ we have

$$(f\cdot g)\cdot h(t, y)=\sum_{t\le x\le y}\Big(\sum_{t\le u\le x} f(t, u)g(u, x)\Big)h(x, y),$$

$$f\cdot(g\cdot h)(t, y)=\sum_{t\le u\le y} f(t, u)\Big(\sum_{u\le x\le y} g(u, x)h(x, y)\Big).$$

Clearly the general term of the same form $f(t,u)g(u,x)h(x,y)$ occurs in each expression. Moreover, for fixed t and y, both the ranges for the variable points u and

x are given by $\{t\le x\le y,\ t\le u\le x\}$ and $\{t\le u\le y,\ u\le x\le y\}$ respectively, which may be reduced to the same range $\{t\le u\le x\le y\}$. Hence $(f\cdot g)\cdot h(x,\ y)=f\cdot(g\cdot h)(x,\ y)$ is justified by the formal equality postulate.

In particular, for a formally computable element f, one can define $f^2=f\cdot f$, $f^{m+1}=f^m\cdot f=f\cdot f^m$, and in addition, define $f^0=\delta$, so that the law of indices $f^m\cdot f^n=f^{m+n}$ holds for $m,\ n\ge 0$. According to non-standard analysis one may even define f^ω by the principle of extension. (Details will be omitted here).

All the formally computable elements (in the general sense) with regard to $O(\omega)$ summations of types (I) and (II) form a subclass $\mathfrak{T}'\subset\mathfrak{T}$. Let f be an element of $\mathfrak{T}'$ such that $f(x,\ x)\neq 0$ for $x\in\hat{S}$. Then by using the principle of transfinite induction one can define a function $g(x,\ y)$ inductively such that $f\cdot g=g\cdot f=\delta$. This is the inverse of f and may be written as $g=f^{-1}$. Thus in conclusion one can get a generalized incidence algebra ${}^*\mathscr{A}(\mathfrak{T}')$ defined on $\mathfrak{T}'\subset(\hat{S}\times\hat{S}\xrightarrow{f}{}^*R)$.

Example 2. Given $x,\ y\in{}^*R$ with $x<y$, $(y-x)/\varepsilon\in{}^*N\setminus N$. Then, as in the ordinary case, $\lambda^k(x,\ y)(k\in{}^*N)$ just represents the number of proper chains of length k that can be stretched from x to y, namely, (cf. [3])

$$\lambda^k(x,\ y)=\binom{(y-x)/\varepsilon-1}{k-1}=\binom{(y-x)\omega-1}{k-1},$$

the right-hand side being the non-standard binomial coefficient defined in ${}^*R$. This is a number in ${}^*N\setminus N$. Evidently we have the standard part

$${}^\circ(\varepsilon^{k-1}\cdot\lambda^k(x,\ y))={}^\circ((y-x)^{k-1}/(k-1)!).$$

In particular one may define $\lambda^m(x,\ y)=0$ when $m>(y-x)/\varepsilon$

Cauchy product rule. Consider $O(\omega)$-summations $\sum_0^{m_1}\alpha_k$ and $\sum_0^{m_2}\beta_k$ where $m_1=O(\omega)$, $m_2=O(\omega)$, and both $\alpha_k\equiv\alpha(k)$ and $\beta_k\equiv\beta(k)$ are formally computable elements. Then with the aid of the formal equality postulate we obtain the formal product rule

$$\left(\sum_{k=0}^{m_1}\alpha_k\right)\left(\sum_{k=0}^{m_2}\beta_k\right)=\sum_{k=0}^{m_3}\gamma_k,\quad \gamma_k=\sum_{i=0}^{k}\alpha_i\beta_{k-i},$$

where $m_3=m_1+m_2=O(\omega)$, $\alpha_i=0\ (i>m_1)$, and $\beta_i=0\ (i>m_2)$. In particular, if $\alpha_k=\alpha(k)$, $\beta_k=\beta(k)$ and $\gamma_k=\gamma(k)$ are finitely computable in ${}^*R$, namely

$${}^\circ\left(\sum_{k=0}^{m_1}\alpha_k\right)\in R,\quad {}^\circ\left(\sum_{k=0}^{m_2}\beta_k\right)\in R,\quad {}^\circ\left(\sum_{k=0}^{m_3}\gamma_k\right)\in R,$$

then the Cauchy rule gives an exact equality after taking standard part on both sides. This follows, of course, from Abel's theorem for power series.

§ 4 General Möbius Function Pairs and Inversion Formulae

We shall say that $\sigma\in$ is an element with a general nilpotent property if for ever given segment $\langle x, y\rangle\in\hat{S}\times\hat{S}$ with card $\langle x, y\rangle=O(\omega)$, there exists a least integer $\nu=\nu(x, y)\in{}^*N$, depending on (x, y) with $y=O(\omega)$, such that $\sigma^m(x, y)=0$ for $m>\nu$. Here the least possible integer $\nu(x, y)$ has the property that $\nu(u, v)\leq\nu(x, y)$ whenever $\langle u, v\rangle\subset\langle x, y\rangle$. A simple example for σ is given by $\sigma=\lambda$. Generally, $\sigma=\sum_{k\geq 1}a_k\lambda^k$ is also nilpotent.

Let $\sum_{k=0}^{\infty}a_kt^k$ and $\sum_{k=0}^{\infty}b_kt^k$ be formal power series with coefficients $a_k, b_k\in{}^*R$ such that (in accordance with the ordinary Cauchy product rule)

$$\left(\sum_{k=0}^{\infty}a_kt^k\right)\left(\sum_{k=0}^{\infty}b_kt^k\right)\equiv a_0b_0,$$

where $^\circ(a_0b_0)=1$. Then $\{a_k\}$ and $\{b_k\}$ are called reciprocal sequences.

Definition 6. Let $\{a_k\}$ and $\{b_k\}$ be a pair of reciprocal sequcnces, and let $\sigma\in\mathfrak{F}'$ be a formally computable element that has a general nilpotent property. Then

$$\mu_1=\sum_{k=0}^{\infty}a_k\sigma^k, \qquad \mu_2=\sum_{k=0}^{\infty}b_k\sigma^k$$

are said to form a general Möbius function pair $\{\mu_1, \mu_2\}$ on $S\times S$, provided that $\mu_1\in\mathfrak{F}'$, $\mu_2\in\mathfrak{F}'$ and that they are finitely computable in $^*R$, namely for any $(x, y)\in\hat{S}\times\hat{S}$

$$^\circ\left(\sum_{x\leq y}\mu_1(x, y)\right)\in R, \quad ^\circ\left(\sum_{x\leq y}\mu_2(x, y)\right)\in R.$$

Note that for the partitcular case that S is locally finite, one can simply define an incidence algebra $\mathscr{A}(\mathfrak{F})$ on $\mathfrak{F}\subset(S\times S\to{}^*R)$. In other words, one may simply use S itself instead of $\hat{S}$, and there is no need to introduce $\mathfrak{F}'$, since the formal equality postulate is always satisfied in this case. Similarly we have a pair of general Möbius functions $\{\mu_1, \mu_2\}$ defined on $S\times S$ (cf. Definition 6). Let us denote

$$\bar{S}:=\begin{cases} S \text{ when S is locally finite},\\ \hat{S} \text{ when S is locally infinite with property } (\omega).\end{cases}$$

Theorem (Generalized Möbius-Rota inversion formulae). Let $S\equiv(S,\leq)$ be a point representable poset which is either locally finite or locally infinite but with property (ω). Then for every general Möbius function pair $\{\mu_1, \mu_2\}$ defined on $\bar{S}\times\bar{S}$, we have the following pair of inversion formulas

(3 . 1) $$°(f(y)) = °\left(\sum_{x\le y}\mu_1(x, y)g(x)\right),$$

(3 . 2) $$°(g(y)) = °\left(\sum_{x\le y}\mu_2(x, y)f(x)\right),$$

provided that both f and g are functions of ($\overline{S}\to{}^*R$) and formally computable in the general sense in $^*R$.

Proof. For the cass S being locally finite, the theorem has already been proved proviously (cf. [1]). Here it suffices to treat the case that S belongs to the category of Definition 3. For any given $(x, y)\in S\times\hat{S}$, there are only a finite number of maximal chains of $\hat{S}$ that may pass through x and y, since all the chains lie on L_i's. Thus by property (ω) it is clear that summations of the types (I) and (II) are always $O(\omega)$-summations.

As σ is formally computable in the general sense, it satisfies the law of indices, so that $\sigma^0=\delta$, $\sigma^m\cdot\sigma^n=\sigma^{m+n}$. We have to verify that $(\mu_1\cdot\mu_2)(t, y)=\delta(t, y)$. By the nilpotent property of σ there exists an $\nu=\nu(t, y)\in{}^*N$ with $\nu=O(\omega)$ such that $\sigma^m(t, y)=0$ for $m>\nu$. Consequently we have $\sigma^m(t, x)=\sigma^m(x, y)=0$ for $m>\nu(t, y)$ when $(t, x)\subset(t, y)$, $(x, y)\subset(t, y)$. Thus by repeated applications of the formal equality postulate we have

$$\begin{aligned}(\mu_1\cdot\mu_2)(t, y) &= \sum_{t\le x\le y}\left\{\sum_{k=0}^{\nu}a_k\sigma^k(t, x)\right\}\left\{\sum_{j=0}^{\nu}b_j\sigma^j(x, y)\right\}\\ &= \sum_{t\le x\le y}\sum_{s=0}^{2\nu}\left\{\sum_{j=0}^{s}a_{s-j}b_j\sigma^{s-j}(t, x)\sigma^j(x, y)\right\}\\ &= \sum_{s=0}^{2\nu}\sum_{j=0}^{s}a_{s-j}b_j\left\{\sum_{t\le x\le y}\sigma^{s-j}(t, x)\sigma^j(x, y)\right\}\\ &= \sum_{s=0}^{2\nu}\left\{\sum_{j=0}^{s}a_{s-j}b_j\right\}\sigma^s(t, y)\\ &= a_0b_0\sigma^0(t, y) = a_0b_0\delta(t, y),\end{aligned}$$

where we have re-defined $a_k=b_k=0$ for those k with $(\nu+1)\le k\le 2\nu$.

Now suppose that (3.1) is given in which $g(x)$ is formally computable. Then, substituting (3.1) into the right-hand side of (3.2), we get

$$°\left(\sum_{x\le y}\mu_2(x, y)f(x)\right) = °\left(\sum_{x\le y}\mu_2(x, y)\left(\sum_{t\le x}\mu_1(t, x)g(t)+\varepsilon_1(x)\right)\right)$$

where $°(\varepsilon_1(x))=0$. Since both μ_1 and μ_2 are formally computable, we can still make use of the formal equality postulate to evaluate the above right hand side as follows

$$\left(\sum_{t\leq y} g(t)\sum_{t\leq x\leq y}\mu_1(t,x)\mu_2(x,y)\right)+{}^{\circ}\left(\sum_{x\leq y}\varepsilon_1(x)\mu_2(x,y)\right)$$

$$={}^{\circ}\left(\sum_{t\leq y} g(t)\mu_1\cdot\mu_2(t,y)\right)={}^{\circ}\left(\sum_{t\leq y} g(t)a_0b_0\ \delta(t,y)\right)={}^{\circ}(a_0b_0g(y))={}^{\circ}(g(y)).$$

Here we have used the fact that ${}^{\circ}(a_0b_0)=1$, ${}^{\circ}(\varepsilon_1(x))=0$ and

$${}^{\circ}\left(\sum_{x\leq y}\mu_2(x,y)\right)\in R.$$

This shows that (3.1) implies (3.2). Similarly (3.1) can also be deduced from (3.2). Hence the theorem is true.

Corollary. Let S be a locally finite poset containing a least element (foremost element). Let $\mu_1=\sum_0^{\omega}a_k\sigma^k$ and $\mu_2=\sum_0^{\omega}b_k\sigma^k$ be a pair of general Möbius functions belonging to $(S\times S\to R)$, so that $\mu_1\cdot\mu_2=\sigma^0=\delta$, where σ is an nilpotent element such that $\sigma^k(x,y)=0$ for $k\geq n(x,y)$, $n(x,y)$ being a positive integer depending upon $(x,y)\in S\times S$. Then we have the reciprocal relations

$$f(y)=\sum_{x\leq y}\mu_1(x,y)g(x), \quad (3.3)$$

$$g(y)=\sum_{x\leq y}\mu_2(x,y)f(x), \quad (3.4)$$

where $y\in S$ and f and g are functions of $(S\to R)$.

Example 3. Let $\sigma=\sum_{k\geq 1}a_k\lambda^k$ be a formal power series in λ (the incidence function). Then we have the reciprocal pair (3.3)-(3.4).

Example 4. Take $\mu_1=\lambda^0+\lambda=\zeta$, $(\lambda^0=\delta)$, so that we have a Möbius function pair $\{\mu_1,\mu_2\}$ with $\mu_2=\zeta^{-1}=(\delta+\lambda)^{-1}=\delta-\lambda+\lambda^2-\lambda^3+\cdots$. This leads to the classical Möbius-Weisner-Hall-Rota inversion formulae (cf. [7]).

Example 5. Let $g(x)$ be contiunous on $S\equiv[0,\infty)\in R$. Suppose that $\mu_1=\varepsilon\zeta=\varepsilon\lambda^0+\varepsilon\lambda$ and $\mu_2=\varepsilon^{-1}\zeta^{-1}$ are defined on $\hat{S}\equiv\{\varepsilon,2\varepsilon,\cdots,n\varepsilon,\cdots\}$, and denote ${}^{\circ}(n\varepsilon)=a$. Then we have

$$f(a)={}^{0}(f(n\varepsilon))={}^{\circ}\left(\sum_{k=1}^{n}\varepsilon\zeta(k\varepsilon,n\varepsilon)g(k\varepsilon)\right)={}^{\circ}\left(\varepsilon\sum_{k=1}^{n}g(k\varepsilon)\right)=\int_0^a g(t)\mathrm{d}t,\quad f(0)=0,$$

$$g(a)={}^{0}(g(n\varepsilon))={}^{\circ}\left(\sum_{k=1}^{n}\varepsilon^{-1}\zeta^{-1}(k\varepsilon,n\varepsilon)f(k\varepsilon)\right)={}^{\circ}\left(f(n\varepsilon)-f((n-1)/\varepsilon)\right)=\frac{\mathrm{d}}{\mathrm{d}a}f(a).$$

This shows that the reciprocity between integration and differentiation is also implied by (3.1)–(3.2).

§ 5 Applications to Integral Equations

First application. Consider Volterra integral equation of the second kind

$$(4.1)\qquad u(x)=f(x)+\rho\int_a^x K(\xi,\ x)u(\xi)d\xi,$$

where $f\in C([a,\ b])$, $K\in C([a,\ b]^2)$, and ρ is a numerical constant. We have natural extensions

$$K\in({}^*[a,\ b]^2\to{}^*\mathrm{R}),\quad f,\ u\in({}^*[a,b]\to{}^*\mathrm{R}).$$

Denote $\mathrm{I}\equiv{}^*[a,\ b]$, and let $\hat{\mathrm{I}}\equiv\hat{\mathrm{I}}_\varepsilon$ be its ε-partitioned structure, namely

$$\hat{\mathrm{I}}:=\{\nu\varepsilon\,|\nu\in{}^*N,\ a\le{}^0(\nu\varepsilon)\le b\}.$$

Certainly one can define an incidence algebra ${}^*\mathscr{A}$ on $\hat{\mathrm{I}}\times\hat{\mathrm{I}}$. Let σ be an element ${}^*\mathscr{A}$ defined by $\sigma=\varepsilon K\equiv\varepsilon K_1$. Then one can introduce the Dedekind product

$$\sigma^2(k\varepsilon,\ n\varepsilon)=(\varepsilon K_1\cdot\varepsilon K_1)(k\varepsilon,\ n\varepsilon)$$

$$=\varepsilon\sum_{k<j<n}\varepsilon K_1(k\varepsilon,\ j\varepsilon)K_1(j\varepsilon,\ n\varepsilon)\overset{\triangle}{=}\varepsilon K_2(k\varepsilon,\ n\varepsilon),$$

where ${}^\circ(k\varepsilon)=a$, ${}^\circ(n\varepsilon)=x$, k, $n\in{}^*\mathrm{N}$, and we have used the summation range $k<j<n$ insted of $k\le j\le n$, since in fact ${}^\circ(k\varepsilon)=((k+1)\varepsilon)=a$, etc. Inductively we have

$$\sigma^{m+1}(k\varepsilon,\ n\varepsilon)=(\varepsilon K_m\cdot\varepsilon K_1)(k\varepsilon,\ n\varepsilon)\overset{\triangle}{=}\varepsilon K_{m+1}(k\varepsilon,\ n\varepsilon)$$

$$=\sum_{k<j_1<j_2\cdots<j_m<n}\sigma(k\varepsilon,\ j_1\varepsilon)\sigma(j_1\varepsilon,\ j_2\varepsilon)\cdots\sigma(j_m\varepsilon,\ n\varepsilon).$$

Thus it follows that σ has a nilpotent property. Indeed, $\sigma^{m+1}(k\varepsilon,\ n\varepsilon)=0$ for $m>(n-k)$.

Now in ${}^*\mathrm{R}$ we may rewrite (4.1) in the form

$${}^\circ(f(x))={}^\circ\Big(u(x)-\rho\int_a^x K(\xi,\ x)u(\xi)d\xi\Big)$$

$$={}^\circ\Big(u(n\varepsilon)-\rho\sum_{k<j<n}\varepsilon K_1(j\varepsilon,\ n\varepsilon)u(j\varepsilon)\Big)$$

$$={}^\circ\Big(\sum_{k<j\le n}\mu_1(j\varepsilon,\ n\varepsilon)u(j\varepsilon)\Big)$$

where $\mu_1=\sigma^0-\rho\sigma$, $\sigma^0=\delta$. Consequently, one may get a pair $\{\mu_1,\ \mu_2\}$ by taking $\mu_2=(\sigma^0-\rho\sigma)^{-1}=\sigma^0+\rho\sigma+\cdots+\rho^k\sigma^k+\cdots,\quad(k\in\mathrm{N})$.

Thus by (3.1)⟺(3.2) we obtain, in accordance with non-standard analysis,

$$(4.2)\qquad {}^\circ(u(x))={}^\circ\Big(\sum_{k<j<n}\mu_2(j\varepsilon,\ n\varepsilon)f(j\varepsilon)\Big)=f(x)+\rho\int_a^x\Gamma(\xi,x,\rho)f(\xi)d\xi,$$

where $\Gamma(\xi, x, \rho)$ is the resolvent kernel and may be written as

$$\Gamma(\xi, x, \rho)=\sum_{n=0}^{\infty} \rho^n K_{n+1}(\xi, x),$$

$K_{n+1} \in([a, b]^2 \to \mathrm{R})$ being the $(n+1)$th iterated kernel.

Evidently (4.2) is just the Liouville-Neumann solution to (4.1). More generally, one may take a pair of general Möbius functions of the forms (with $\sigma^0 = \delta$)

$$(4.3) \qquad \mu_1=\sigma^0+\sum_{k\geq 1} a_k(\rho\sigma)^k, \qquad \mu_2=\sigma^0+\sum_{k\geq 1} b_k(\rho\sigma)^k,$$

where $\{a_k\}$ and $\{b_k\}$ are reciprocal sequences in R with $a_0 b_0=1$. Then a general proposition may may be stated as follows.

Proposition 1. Let $K(\xi, x) \in \mathrm{C}([a, b]^2)$ with Chebyshev norm $\|K\| \leq M$. For given parameter $\rho \in \mathrm{R}$, let $\{a_k\}$ and $\{b_k\}$ be reciprocal sepuences in R such that

$$(4.4) \qquad \sum_{n=0}^{\infty} \frac{1}{n!}(\rho M(b-a))^n \cdot \max\{|a_n|, |b_n|\}<\infty.$$

Define a pair of generalized Neumaun series in terms of iterated kernels:

$$(4.5) \qquad G(\xi, x, \rho)=\sum_{n=0}^{\infty} \rho^n a_{n+1} K_{n+1}(\xi, x),$$

$$(4.6) \qquad F(\xi, x, \rho)=\sum_{n=0}^{\infty} \rho^n b_{n+1} K_{n+1}(\xi, x),$$

Then there exist a pair of reciprocal integral transforms as follows

$$(4.7) \qquad f(x)=g(x)+\rho \int_a^x G(\xi, x, \rho) g(\xi) d \xi,$$

$$(4.8) \qquad g(x)=f(x)+\rho \int_a^x F(\xi, x, \rho) f(\xi) d \xi,$$

provided that $f, g \in \mathrm{C}([a, b])$. Generally, (4.8) gives the unique solution to the integral equation (4.7), and vice versa.

Apparently the forms of (4.5) and (4.6) are suggested by the Möbius function pair $\{\mu_1, \mu_2\}$, and the relation $(4.7) \Longleftrightarrow (4.8)$ follows from the inversion theorem $(3.1) \Longleftrightarrow (3.2)$ in which all the $O(\omega)$-summations have been transformed into integrals with variable upper limits by the aid of non-standard analysis. As may easily be verified, the condition (4.4) for reciprocal sequences sufficiently ensures the validity of the limiting processes involved in (4.7)-(4.8).

Second application. Let $\hat{S}$ be the ε-partitioned structure of $S \equiv [0, \infty) \in {}^*\mathrm{R}$, and let $\{\mu_1, \mu_2\}$ be defined by

$$\mu_1 = \sum_{k \geq 0} a_k \varepsilon^k \lambda^k, \qquad \mu_2 = \sum_{k \geq 0} b_k \varepsilon^k \lambda^k,$$

where $\{a_k\}$ and $\{b_k\}$ are reciprocal sequences in R with $a_0 = b_0 = 1$, and λ is the incidence function defined on $\hat{S} \times \hat{S}$. Then an application of (3.1)—(3.2) to g, $f \in C(S)$ in R yields the following

$$f(a) = g((a) + \sum_{k \geq 1} \frac{a_k}{(k-1)!} \int_0^a (a-t)^{k-1} g(t) \mathrm{d}t, \tag{4.9}$$

$$g(a) = f(a) + \sum_{k \geq 1} \frac{b_k}{(k-1)!} \int_0^a (a-t)^{k-1} f(t) \mathrm{d}t. \tag{4.10}$$

These are a kind of specialization of (4.7)—(4.8). As shown in [2], one may make use of (4.9)—(4.10) to obtain solutions to convolution integral equations, namely

Assertion 1. Let $K(z)$ and $H(z)$ be analytic for $\mathrm{Re}(z) \geq 0$ such that $\{1, K(0), K'(0), \cdots\}$ and $\{1, H(0), H'(0), \cdots\}$ just form a pair of reciprocal sequences in R. Then we have the following pair of convolution integral transforms

$$f(a) = g(a) + \int_0^a K(a-t) g(t) \mathrm{d}t, \tag{4.11}$$

$$g(a) = f(a) + \int_0^a H(a-t) f(t) \mathrm{d}t, \tag{4.12}$$

provided that both the integrals exist in the sense of Cauchy's integration.

Assertion 2. Lel $K(z)$ and $H(z)$ be analytic for $\mathrm{Re}(z) \geq 0$ with $K^{(r)}(o) = 1$ $(r \geq 0)$ and $K(0) = \cdots = K^{(r-1)}(0) = 0$ when $r \geq 1$. Suppose that $\{1, H(0), H'(0), \cdots\}$ and $\{K^{(r)}(0), K^{(r+1)}(0), \cdots\}$ form a pair of reciprocal sequences. Then then the convolution integral equation

$$g(a) = \int_0^a K(a-t) f(t) \mathrm{d}t \tag{4.13}$$

with the boundary condition $g(0) = \cdots = g^{(r)}(0) = 0$, $g^{(r+1)}(0) \neq 0$ has a solution of the form

$$f(a) = g^{(r+1)}(a) + \int_0^a H(a-t) g^{(r+1)}(t) \mathrm{d}t. \tag{4.14}$$

Of course, (4.11)$\Longleftrightarrow$(4.12) and (4.13)$\Rightarrow$(4.14) may also be verified by classical methods. Now using the condition of Assertion 1 we have

$$\int_0^\infty e^{-s} K(ts) \mathrm{d}s = \sum_{r=0}^{\infty} K^{(r)}(0) t^r$$

provided that the series converes when t near zero. Conequently we obtain

$$\left(1 + t\int_0^\infty e^{-s} K(ts) \mathrm{d}s\right)^{-1} = 1 + \sum_{r=0}^{\infty} H^{(r)}(0) t^{r+1}.$$

Hence $H(z)$ can be given an analytic representation, viz.

$$(4.15) \qquad H(z)=\sum_{n=1}^{\infty}\frac{z^{n-1}}{(n-1)!n!}\left(\frac{d}{dt}\right)_0^n\left(1+t\int_0^{\infty}e^{-s}K(ts)ds\right)^{-1}.$$

Similarly $K(z)$ may be written as

$$(4.16) \qquad K(z)=\sum_{n=1}^{\infty}\frac{z^{n-1}}{(n-1)!n!}\left(\frac{d}{dt}\right)_0^n\left(1+t\int_0^{\infty}e^{-s}H(ts)ds\right)^{-1}.$$

Clearly (4.15)-(4.16) form a pair of reciprocal relations. Consequently Assertion 1 leads to the following

Proposition 2. The convolution integral equation (4.11) has an explicit solution (4.12) in which $H(z)$ takes the from (4.15). Similarly (4.11) and (4.16) give an explicit solution to (4.12).

In a like manner one may deduce from Assertion 2 the following

Proposition 3. Let $K(z)$ satisfy the condition of Assertion 2. Then the integral equation (4.13) wtth the boundary condition $g(0)=\cdots=g^{(r)}(0)=0$, $g^{(r+1)}(0)\neq 0$ has a solution of the form (4.14) in which $H(z)$ may be written explicitly in the form

$$(4.17) \qquad H(z)=\sum_{n=1}^{\infty}\frac{z^{n-1}}{(n-1)!n!}\left(\frac{d}{dt}\right)_0^n\left(\int_0^{\infty}e^{-s}K^{(r)}(ts)ds\right)^{-1},$$

provided that $H(z)$ is analytic for $\mathrm{Re}(z)\geq 0$.

Surely these propositions can also be established by means of the Laplace transfrom method or by other methods. Here we just illustrate that they may be deduced from the common origin (3.1)$\Longleftrightarrow$(3.2). As may easily be verified, quite a number of inversion formulas displayed in the inversion tables of the book by H. M. Srivastava and R. G. Buschman ([9], 1977) are implied by the above propositions as special examples.

§ 6 Application to Interpolation Process

What we will describe here is just an illustration that certain explicit forms of polynomial interpolatory splines can be determined with the aid of a special type of Möbius inversion. The basic result has been established in our provious paper [4].

Let S be a locally finite totally ordered set having a least element. Then every segment $\langle x, y\rangle$ is a chain of finite length, denoted by $r[x, y]$. For an incidence algebra $\mathscr{A}$ defined on $S\times S\to R$, we have

$$\lambda^k(x, y)=\binom{r[x, y]-1}{k-1}$$

Given a Möbius function pair $\mu_1=\sum_{k\geq 0}a_k\lambda^k$ and $\mu_2=\sum_{k\geq 0}b_k\lambda^k$. Then (3.1)—(3.2) imply the following

(5.1) $$f(y)=\sum_{x\le y}\left\{\sum_{k\ge 0}\binom{r[x,\ y]-1}{k-1}a_k\right\}g(x),$$

(5.2) $$g(y)=\sum_{x\le y}\left\{\sum_{k\ge 0}\binom{r[x,\ y]-1}{k-1}b_k\right\}f(x).$$

In particular, if S is taken to be N^+, then (5.1)—(5.2) yield the reciprocal relations

(5.3) $$f(n)=\sum_{k=1}^{n}\left\{\sum_{r=0}^{n-k}\binom{n-k-1}{r-1}a_r\right\}g(k),$$

(5.4) $$g(n)=\sum_{k=1}^{n}\left\{\sum_{r=0}^{n-k}\binom{n-k-1}{r-1}b_r\right\}f(k),$$

where $f,\ g\in(N^+\to R)$, and the usual convention for binomial confficients $\binom{0}{0}=\binom{-1}{-1}=1,\ \binom{-1}{n}=\binom{n}{-1}=0\ (n\neq -1)$ has been assumed.

As was shown in the joint paper by J.X. Yang and the author [3], one may obtain a simpler interpolation formula $\Phi(f;\ \cdot)$ from (5.3)—(5.4) by simply taking

$$a_r=\binom{a}{r},\qquad b_r=\binom{-a}{r},\quad (r=0,\ 1,\ 2,\ \cdots)$$

namely

(5.5) $$\Phi(f;\ x)=\sum_{k=1}^{[x]+a}\binom{x+a-k}{a-1}\triangle^a f(k-a-1),$$

where $a\ge 2$ is a fixed integer parameter, $[x]$ denotes the integer part of $x\ge 0$, and $\triangle$ is the forward difference operator with unit increment. More precisely, $\Phi(f;x)$ is an interpolation formula for any empirical function $f(n)$ defined for $n=0,\ 1,\ 2,\ \cdots$, with supplemental definition $f(-m)=0,\ (m=1,\ 2,\ 3,\ \cdots)$. In fact, it can be verified that $\Phi(f;\ x)\in C([0,\ \infty))$, $S(f,\ n)=f(n)$, $(n=0,\ 1,\ 2,\ \cdots)$, and that for every polynomial $p(x)$ of degree $\le a-1$ defined for $x\ge 0$ with the supplemental definition $p(-m)=0$, $(m=1,2,\cdots)$, one always has $\Phi(p;\ x)\equiv p(x)$, $(0\le x<\infty)$.

Note that $\Phi(f;\ x)$ is not smooth at the data points $x=n,\ n\in N^+$:

$$\Phi'(f;\ n+0)-\Phi'(f;\ n-0)=\frac{1}{a-1}(-1)^a\triangle^a f(n-1).$$

However, a suitable correction term $\Psi(f;\ x)$ may be introduced by means of the two-point Hermite interpolation formula so that $\Phi(f;\ x)+\Psi(f;\ x)$ be comes smooth on $[0, \infty)$. In fact, we have with $a = 2m+2$,

$$(5.6)\qquad \Psi(f;\ x) = (x-[x]-1)^{m+1}\sum_{k=1}^{m} A_k (x-[x])^k \{(x-[x]-1)^{-m-1}\}^{(m-k)}_{x=[x]+0}$$

$$+ (x-[x])^{m+1}\sum_{k=1}^{m} A_k (x-[x]-1)^k \{(x-[x])^{-m-1}\}^{(m-k)}_{x=[x+1]-0},$$

where $\{\phi(x)\}^{(r)}_{x=a}$ denotes Taylor's $(r+1)$ term expansion of $\phi(x)$ in terms of $(x-a)^i$, $(0 \le i \le r)$, and A_k is defined by

$$A_k = \frac{-1}{(2m+1)!} S_1(2m+1,\ k) \triangle^{2m+1} f([x])$$

in which $S_1(n,\ k)$ $(n \ge k)$ denotes the Stirling number of the first kind. Moreover, the summation given by (5.5) (with $a = 2m+2$) can be shortened to the form (cf. [3])

$$(5.7)\qquad \Phi(f;\ x) = \sum_{k=1}^{2m+2} \binom{x-[x]+2m+2-k}{2m+1} \triangle^{2m+2} f_x([x]+k-2m-3),$$

where $f_x(u)$ is defined by $f_x(u) = f(u)$ for $u \ge [x]$; and $f_x(u) = 0$ for $u < [x]$. Then combining (5.6) and (5.7) we get a smooth interpolation formula of the form

$$(5.8)\qquad T(f;\ x) = \Phi(f;\ x) + \Psi(f;\ x)$$

which possesses the properties:

1° $T(f;\ n) = f(n)$, $(n = 0, 1, 2, \cdots)$;

2° $T(f;\ x)$ is a piecewise polynomial of degree $\le 2m+1$ on each interval $[n, n+1)$;

3° For $j = 1, 2, \cdots, m$, we have $(d/dx)^j T(f;\ n+0) = (d/dx)^j T(f;\ n-0)$, $n \in N^+$.

Thus in conclusion we have the following

Proposition 4. For any given sequence $\{f(n)\}_{n=0}^{\infty}$ of data with a supplemental definition $f(-j) = 0$ $(j = 1, 2, \cdots)$, there always exists an explicit smooth interpolation formula $T(f;\ x)$ of the form (5.8) such that it enjoys the properties 1°, 2°, and 3°.

Clearly $T(f;\ x)$ may be regarded as a special kind of polynomial interpolatory spline with degree $2m+1$ and smoothness order m(cf. L.L. Schumaker [8]). of course, $T(f;\ x)$ can also be constructed by some general methods for spline functions.

Example 6. As a very simple instance take $m = 1$. Recall the Stirling number $S_1(3,\ 1) = 2$, and notice that

$$\{(x-[x]-1)^{-2}\}^{(0)}_{x=[x]} = 1 \qquad \{(x-[x])^{-2}\}^{(0)}_{x=[x]+1} = 1.$$

Then by (5.8) we obtain

$$T(f;\ x)=\sum_{k=1}^{4}\binom{x-[x]+4-k}{3}\triangle^4 f_x([x]+k-5)$$
$$-\frac{2}{3}(x-[x])(x-[x]-\frac{1}{2})(x-[x]-1)\triangle^3 f([x]).$$

This is an interpolatory cubic spline of smoothness order one on the interval $[0,\infty)$ with $T(f;\ n)=f(n)$, $(n=0,1,2,\cdots)$.

Remark 1. The smooth interpolation (5.8) possesses a kind of weak extreme property. For instance, for $m=1$, if a function class G is defined by

$G:=\{g\,|g\in C^2[n,\ n+1),\ g(n)=f(n),\ g'(n)=T'(f;\ n),\ n=0,1,2,\cdots\}$,

then it can be proved that for all $g\in G$ we have the inequality

$$\int_0^m [T''(f;\ x)]^2 dx\le\int_0^m [g''(x)]^2 dx,\quad (m\in N^+).$$

Moreover, the equality sign holds only when $g(x)\equiv T(f;\ x)$.

Remark 2. One may write Φ_x, Ψ_x and T_x to indicate that they are applied with regard to x. Let D be a bounded region contained in $[0,\infty)\times[0,\infty)$, and let $\{f(i,\ j)\}$ be a given double sequence of data with the supplemental condition that $f(i,\ j)=0$ for all those $(i,\ j)\notin D$. Then the tensor product form

$$T(f,\ x,\ y)=(\Phi_x\Phi_y+\Psi_x\Psi_y+\Phi_x\Psi_y+\Psi_x\Phi_y)(f;\ x,\ y)$$

yields an explicit interpolation operator for $f(x,y)$ that possesses the properties: (I) $T(f;\ i,\ j)=f(i,\ j)$, $(i,\ j)\in D$; (II) $T(f;\ x,\ y)$ is a piecewise bivariate polynomal on each sequare $[i,\ i+1)\times[j,\ j+1)$, $(i,\ j=0,1,2,\cdots)$ with degrees $2m+1$ in x, y respectively; (III) $(\partial/\partial x)^r(\partial/\partial y)^s T(f;\ x,\ y)$ $(1\le r\le m,\ 1\le s\le m)$ have the same limit values when $(x,\ y)$ tends to the boundary lines $x=i$ and $y=j$ $(i,\ j\in N^+)$ from both sides. (cf. also [4].)

References

[1] L. C. Hsu, An extension of the Möbius-Rota inversion theory with applications, (Chinese, English summary), J. Math. Res. & Exposition, Initial issue (1981), 101—112.

[2] L. C. Hsu, Generalized Möbius-Rota inversion theory associated with non-standard analysis, Science Exploration, Vol. 3(1983), No. 1, 1-8.

[3] L.C. Hsu and J. X. Yang, On a method of constructing interpolation formulas via inverse series relations, J. Math. Res. & Exposition, 2(1982), 113-126.

[4] L.C. Hsu and J. X. Yang, A type of explicit smooth interpolation formula with application to a surface interpolation problem, (Chinese, English summary), J. Numerical Mathematics, A Jour. of Chinese Universities, (1984), No. 4, 355-361.

[5] A. H. Lightstone and A. Robinson, Non-Archimedean Fields and Asymptotic Expansions, North Holland, New York, 1975.

[6] A. Robinson, Non-Standrd Analysis, Amsterdam, 1966.

[7] G. C. Rota, On the foundations of combinatorial theory, Z. Wahrscheinlichkeitstheorie, 2(1964),

340-368.
〔8〕L. L. Schumaker, Spline Functions: Basic Theory, John Wiley & Sons, New York, London, Sydney, Toronto, 1981.
〔9〕H. M. Srivastava and R. G. Buschman, Convolution Integral Equations, John Wiley & Sons, 1977.
〔10〕K. D. Stroyan and W. A. J. Luxemburg, Introduction to the Theory of Infinitesimals, Academic Press, New York, London, 1976.