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A Pointwise "o" Saturation Theorem*

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1. Suppose $d\mu_\rho(\rho \in \Omega)$ is a non-negative Borel measure on $[-\pi, \pi]$ with

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\mu_\rho(t) = 1$$

For $k \in N$, denote

$$a_{k\rho} = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos kx \ d\mu_\rho(x),$$

$$b_{k\rho} = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin kx \ d\mu_\rho(x).$$

We introduce the positive convolution operator L_ρ as follows: for $f \in C_{2\pi}$, we have

$$L_\rho(f, x) = (f * d\mu_\rho)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, d\mu_\rho(t).$$

In 1971, R. A. DeVore ([1]) showed that for non-negative even Borel measure $d\mu_\rho$ and $k \in N$,

$$\lim_{\rho \to \rho_0} \frac{1 - a_{k\rho}}{1 - a_{1\rho}} = \psi_k > 0 .$$

If $f \in C_{2\pi}$, then for each $x \in [-\pi, \pi]$,

$$f(x) - L_\rho(f, x) = o_x(1 - a_{1\rho}), \quad (\rho \to \rho_0)$$

iff $f \equiv$ const.

In this paper, we give a general pointwise "o" saturation theorem for positive convolution operator (not necessarily even). As application, we also give a pointwise Curtis theorem([2]).

2. Suppose $\varphi_\rho \to 0^+ (\rho \to \rho_0)$. Let x be a point in $[-\pi, \pi]$, such that for each neighbourhood U_x of x, we have

$$\int_{U_x} d\mu_\rho(t) \neq o_x(\varphi_\rho), \quad (\rho \to \rho_0),$$

then we say that x is an essential point. Otherwise, x is a negligible point. Denote $M = \max_{|t| \le \pi} f(t)$. When $x_0 \in [-\pi, \pi]$ with $f(x_0) = M$, we let

$$\mathfrak{M}(x_0) = \{t \mid t \text{ is an essential point and } f(x_0 + t) = M\},$$

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$$\mathfrak{M} = \bigcap_{\{x_0 \mid f(x_0) = M\}} \mathfrak{M}(x_0)$$

We need two following lemmas (see[1]).

Lemma 1 For each $x \in [-\pi, \pi]$ let

$$f(x) - L_\rho(f, x) = o_x(\varphi_\rho) \quad (\rho \to \rho_0),$$

If $x \notin \mathfrak{M}$, then x is a negligible point.

Proof. Suppose $x \notin \mathfrak{M}$, there exists $x_0 \in [-\pi, \pi]$ with $f(x_0) = M$, such that $x \notin \mathfrak{M}(x_0)$. Then eitst x is negligible point or $f(x_0 + x) < M$. In the latter case, let

$$U_x = \{t \mid f(x_0 + t) < \frac{1}{2}(M + f(x_0 + x))\},$$

U_x is a neighbourhood of x and

$$\frac{1}{2\pi}(M - f(x_0 + x))\int_{U_x} d\mu_\rho(t) \leqslant \frac{1}{\pi}\int_{U_x} (f(x_0) - f(x_0 + t))d\mu_\rho(t)$$

$$\leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x_0) - f(x_0 + t))d\mu_\rho(t) = f(x_0) - L_\rho(f, x) = o_{x_0}(\varphi_\rho).$$

Since x_0 is depend on x, we have

$$\int_{U_x} d\mu_\rho(t) = o_x(\varphi_\rho), \quad (\rho \to \rho_0),$$

and then x is negligible point.

Lemma 2. If for each $x \in [-\pi, \pi]$

$$f(x) - L_\rho(f, x) = o_x(\varphi_\rho), \quad (\rho \to \rho_0)$$

and $f \not\equiv$ const, then $\mathfrak{M}$ has only a finite number of points. Furthermore, if x is any point in $\mathfrak{M}$, then $x = 2\pi a$, where a is rational.

Proof (see[1]).

It is induced by the lemma 2 that if for each $x \in [-\pi, \pi]$

$$l(x) - L_\rho(f, x) = o_x(\varphi_\rho)$$

and $f \not\equiv$ const, then there is $m \in \mathbb{N}$, sueh that

$$\mathfrak{M} \subseteq \{\frac{k\pi}{m} \mid k = 0, \pm 1, \pm 2, \cdots\}$$

Let

$$I = \bigcup_{k=-m}^{m} [\frac{k\pi}{m} - \frac{\pi}{8m}, \frac{k\pi}{m} + \frac{\pi}{8m}] \cap [-\pi, \pi],$$

$$S = [-\pi, \pi] \setminus I.$$

Since eaeh point $x \in S$ is negligible, we may use a compactness argument to show that

$$\int_S d\mu_\rho(t) = o(\varphi_\rho), \quad (\rho \to \rho_0).$$

3. Using the above fact, we can prove the following pointwise "o" saturation theorem.

Theorem 1. Let $\{L_\rho\}_{\rho\in\Omega}$ be a sequence of positive convolution operators (not necessarily even), Suppose that for $\forall\ k\in \mathrm{N}$ we have

$$\lim_{\rho\to\rho_0}\frac{|1-a_{2k,\rho}-|b_{2k,\rho}||}{\varphi_\rho}=\psi_k>0 .$$

If $f\in C_{2\pi}$, then for each $x\in[-\pi,\pi]$

$$f(x)-L_\rho(f,x)=o_x(\varphi_\rho),\quad(\rho\to\rho_0)$$

iff $f\equiv$const.

Proof The "if" part of theorem is obvious. The proof of the "only if" part is based on a trigonometric analogue of parabola technique of Bajsaski-Bojanic ([3]). Since $f\in C_{2\pi}$ and for each $x\in[-\pi,\pi]$

$$f(x)-L_\rho(f,x)=o_x(\varphi_\rho),\quad(\rho\to\rho_0),$$

but $f\not\equiv$const., by subtracting a constant, if necessary, we can suppose that $f(-\pi)=f(\pi)=0$ and $M=\max\limits_{|x|\leq\pi} f(x)>0$. We consider two cases as follows:

When $1-a_{2m,\rho}>|b_{2m,\rho}|$, set

$$h(x)=-M\sin^2 mx+2M.$$

Then $h(x)\geq f(x)$ on $[-\pi,\pi]$. Let

$$C=\min_{t\in I}(h(t)-f(x))>0 .$$

Then $h(x)-C\geq f(x)$ on I, and for some $y\in$ I

$$h(y)-C=f(y).$$

Since

$$h(x)-h(y)=-M\cos2my\sin^2 m(x+y)-\frac{M}{2}\sin2my\sin2m(x-y),$$

then

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(h(x)-h(y))\,d\mu_\rho(x-y)=-\frac{M}{2}(1-a_{2m,\rho})\cos2my+b_{2m,\rho}\sin2my).$$

Because $\cos2my>0$ as $y\in$ I, hence

$$(1-a_{2m,\rho})\cos2my+b_{2m,\rho}\sin2my>\frac{\sqrt{2}}{2}(1-a_{2m,\rho}-|b_{2m,\rho}|)>0$$

and then

$$\int_{-\pi}^{\pi}(h(x)-h(y))\,d\mu_\rho(x-y)\leq-\frac{\sqrt{2}}{4}M(1-a_{2m,\rho}-|b_{2m,\rho}|)<0 .$$

By lemma 2,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(h(x)-h(y))\,d\mu_\rho(x-y)=\frac{1}{\pi}\int_{-\pi}^{\pi}(h(x)-h(y))\,d\mu_\rho(x-y)-\frac{1}{\pi}\int_{S}(h(x)-h(y))\,d\mu_\rho(x-y)\leq-\frac{\sqrt{2}}{4}M(1-a_{2m,\rho}-|b_{2m,\rho}|)+o(\varphi_\rho).$$

Since $f(x)-f(y)\leqslant h(x)-h(y)$ on I, we have

$$L_\rho(f,y)-f(y)=\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x)-f(y))\,\mathrm{d}\mu_\rho(x-y)=\frac{1}{\pi}\int_{I}(f(x)-f(y))\,\mathrm{d}\mu_\rho(x-y)+o(\varphi_\rho)\leqslant\frac{1}{\pi}\int_{I}(h(x)-h(y))\,\mathrm{d}\mu_\rho(x-y)+o(\varphi_\rho)$$

$$\leqslant-\frac{M}{4}\sqrt{2}\,(1-a_{2m,\rho}-|b_{2m,\rho}|)+o(\varphi_\rho).$$

Hence

$$\lim_{\rho\to\rho_0}\frac{|L_\rho(f,y)-f(y)|}{\varphi_\rho}\geqslant\frac{\sqrt{2}}{4}M\psi_m>0.$$

When $|b_{2m,\rho}|>1-a_{2m,\rho}>0$, set

$$h(x)=-M\,\mathrm{sgn}b_{2m,\rho}\sin 2mx+2M.$$

Then $h(x)\geqslant f(x)$ on $[-\pi,\pi]$. Note that

$$h(x)-h(y)=-M\,\mathrm{sgn}\,b_{2m,\rho}(\cos 2my\sin 2m(x-y)-2\sin^2 m(x-y)),$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(h(x)-h(y))\,\mathrm{d}\mu_\rho(x-y)$$

$$=-M(|b_{2m,\rho}|\cos 2my-\mathrm{sgn}b_{2m,\rho}\sin 2my(1-a_{2m,\rho}))$$

$$\leqslant-\frac{\sqrt{2}}{2}M(|b_{2m,\rho}|-1-a_{2m,\rho}))\quad 0.$$

Similarly, we have

$$L_\rho(f,y)-f(y)=\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x)-f(y))\,\mathrm{d}\mu_\rho(x-y)\leqslant-\frac{\sqrt{2}}{2}M(|b_{2m,\rho}|-(1-a_{2m,\rho}))+o(\varphi_\rho),$$

therefore

$$\lim_{\rho\to\rho_0}\frac{|L_\rho(f,y)-f(y)|}{\varphi_\rho}\geqslant\frac{\sqrt{2}}{2}M\psi_m>0.$$

As the conclusion of the above dicussions, we must have $f(y)-L_\rho(f,y)\neq o_y(\varphi_\rho)$, which is the desirable contradiction and therefore $M=0$. This shows that $f(x)\leqslant 0$ on $[-\pi,\pi]$. To show that $f(x)\geqslant 0$ on $[-\pi,\pi]$, we may consider $-f$ insfead of f in the adove argument, and complete the proof of the theorem.

Specially, if $\mathrm{d}\mu_\rho$ is even and $\varphi_\rho=1-a_{1\rho}$, then theorem 1 is the pointwise "o" saturation theorem of Devore([1]) .

It is evident from the proof of theorem 1, that the limited condition of theorem 1 may be replaced by the following weaker condition, i.e., for $\forall k\in\mathrm{N}$,

$$\lim_{\rho\to\rho_0}\frac{|1-a_{2k,\rho}-|b_{2k,\rho}||}{\varphi_\rho}=\psi_k>0.$$

We consider two example as follows.

Example 1 For $f\in C_{2\pi}$, let

$$A_n(f,x)=\int_{-\pi}^{\pi} f(x+t)\,d\delta_{\frac{1}{n}}(t),$$

where $d\delta$ is a Dirac measure. Then for $k\in N$, we have

$$1-a_{kn}=1-\cos\frac{k}{n},\quad b_{kn}=\sin\frac{k}{n}.$$

Taking $\varphi_n=b_{1n}\sim\frac{1}{n}$, we have

$$\lim_{n\to\infty}\frac{|1-a_{2m,n}-|b_{2m,n}||}{\varphi_n}=2m$$

It follows from theorem 1 that, if $f\in C_{2\pi}$ and for each $x\in[-\pi,\pi]$

$$f(x)-A_n(f,x)=f(x)-f(x+\frac{1}{n})=o_x(\frac{1}{n}),$$

then $f(x)\equiv$const.

Example 2 Suppose

$$d\mu_n(t)=(1-\frac{1}{\sqrt{n}})\,d\delta_{\frac{1}{n}}(t)+\frac{\pi}{\sqrt{n}}\,d\delta_{\frac{\pi}{2}}(t).$$

Let for $f\in C_{2\pi}$,

$$\overline{A}_n(f,x)=(f*d\mu_n)(x)$$

Then for $k\in N$

$$1-a_{kn}=2\left((1-\frac{1}{\sqrt{n}})\sin^2\frac{k}{2n}+\frac{1}{\sqrt{n}}\sin^2\frac{k\pi}{4}\right),$$

$$b_{kn}=(1-\frac{1}{\sqrt{n}})\sin\frac{k}{n}+\frac{1}{\sqrt{n}}\sin\frac{k\pi}{2}$$

Taking $\varphi_n=b_{2n}\sim\frac{2}{n}$, we have

$$\lim_{n\to\infty}\frac{|1-a_{2m,n}-|b_{2m,n}||}{\varphi_n}==\begin{cases}2s & m=2s,\\ +\infty & m=2s-1.\end{cases}$$

It follows from theorem 1 that, if $f\in C_{2\pi}$ and $f(x)-\overline{A}_n(f,x)=o_x(\frac{1}{n})$, for each $x\in[-\pi,\pi]$, then $f(x)\equiv$const.

4. Now, suppose $d\mu_n(t)=T_n(t)\,dt$, where $T_n(t)$ is a non-negative trigonometric polynomial of degree n with

$$\frac{1}{\pi}\int_{-\pi}^{\pi}T_n(t)\,dt=1.$$

Writing

$$1-a_{kn}=\frac{2}{\pi}\int_{-\pi}^{\pi}\sin^2\frac{kt}{2}T_n(t)\,dt,\quad b_{kn}=\frac{1}{\pi}\int_{-\pi}^{\pi}\sin kt\,T_n(t)\,dt.$$

For $f \in C_{2\pi}$, let

$$\varphi_n(f, x) = (f * T_n)(x),$$

that is called positive trigonometric convolution operator. In 1965, Curtis ([2]) have shown that $T_n(t)$ is non-negative, even trigonometric. If $f \in C_{2\pi}$ and

$$\lim_{n \to \infty} n^2 \| f - \varphi_n(f) \|_{C_{2\pi}} = 0,$$

then $f = \text{const}$.

Based on Theorem 1 and the following Curtis lemma: There is a $C_0 > 0$, such that for any integers and k with $n \geq k \geq 1$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2\frac{kt}{2} T_n(t)\, dt \geq C_0 \frac{k^2}{n^2}.$$

We give following pointwise Curtis Theorem.

Theorem 2 Let $\{\varphi_n\}_{n \in N}$ be a sequence of positive trigonometric convolution operators (not necessarily even) and $b_{2k,n} = o(n^{-2})$, for $\forall\ k \in N$.

If $f \in C_{2\pi}$ and $\lim_{n \to \infty} n^2 | f(x) - \varphi_n(f, x) | = 0$, for each $x \in (-\pi, \pi]$, then $f = \text{const}$.

Proof For any integers $n \geq k \geq 1$, it bollows from the Curtis lemma that

$$1 - a_{k n} = \frac{2}{\pi}\int_{-\pi}^{\pi} \sin^2 \frac{kt}{2} T_n(t)\, dt \geq 2 C_0 \frac{k^2}{n^2}.$$

Take $\varphi_n = n^{-2}$. Then for $\forall\ k \in N$ and $n \geq k \geq 1$.

$$\frac{1 - a_{2k,n}}{\varphi_n} \geq 8 C_0 k^2.$$

Hence

$$\psi_m = \lim_{n \to \infty} \frac{| 1 - a_{2k,n} - | b_{2k,n} | |}{\varphi_n} \geq 8 C_0 k^2 > 0.$$

By theorem 1, this complete the proof of the Theorem 2.

Reference

[1] DeVore, R. A., Linear Operators and Approximation, 1971, 364—370.

[2] Curtis, P. C., Mich. Math. J. 12, 1965, 155—160.

[3] DeVore, R. A., The Approximation of Continuous Functions by Positive Linear Operators, Lecture notes in Maths. 293, 101—105