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Rearrangement Inequalities*

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I. Introduction

Let $(a_{1j}, a_{2j}, \cdots, a_{t_j j})$ ($1 \le j \le k$) be sequences of length t_j, where $a_{ij} \ge 0$ and $n = t_1 + t_2 + \cdots + t_k$. Let $a'_1, a'_2, \cdots, a'_n$ be the a_{ij} ($1 \le i \le t_j$; $1 \le j \le k$) arranged in non-decreasing order: $a'_1 \le a'_2 \le \cdots \le a'_n$; and $a^*_1, a^*_2, \cdots, a^*_n$ be the a_{ij} ($1 \le i \le t_j$; $1 \le j \le k$) arranged in non-increasing order: $a^*_1 \ge a^*_2 \ge \cdots \ge a^*_n$. We also write

$$(a^*_1, a^*_2, \cdots, a^*_n) = (a^*_{11}, a^*_{21}, \cdots, a^*_{t_1 1}, \cdots, a^*_{1k}, a^*_{2k}, \cdots, a^*_{t_k k}), \tag{1.1}$$

$$(a'_1, a'_2, \cdots, a'_n) = (a'_{11}, a'_{21}, \cdots, a'_{t_1 1}, \cdots, a'_{1k}, a'_{2k}, \cdots, a'_{t_k k}). \tag{1.2}$$

In [1], H.Mine established the following rearrangement inequality:

$$\prod_{j=1}^{k} (1 + a_{1j} a_{2j} \cdots a_{t_j j}) \le \begin{cases} \prod\limits_{i=1}^{\frac{n}{2}} (1 + a^*_{2i-1} a^*_{2i}), & \text{if } 2 \mid n \\ \prod\limits_{i=1}^{(n-3)/2} (1 + a^*_{2i-1} a^*_{2i})(1 + a^*_{n-2} a^*_{n-1} a^*_n), & \text{if } 2 \nmid n. \end{cases} \tag{1.3}$$

Later he generalized in [2] the result, and proved the following theorems.

Theorem 1 Let $t_1 \ge t_2 \ge \cdots \ge t_k \ge 1$. If $a_{ij} \le 1$ ($1 \le i \le t_j$; $1 \le j \le k$), then

$$\sum_{j=1}^{k} \prod_{i=1}^{t_j} a_{ij} \le \sum_{j=1}^{k} \prod_{i=1}^{t_j} a'_{ij} . \tag{1.4}$$

If $a_{ij} \ge 1$ ($1 \le i \le t_j$; $1 \le j \le k$), then

$$\sum_{j=1}^{k} \prod_{i=1}^{t_j} a_{ij} \le \sum_{j=1}^{k} \prod_{i=1}^{t_j} a^*_{ij}. \tag{1.5}$$

Theorem 2 Let $t_1 \ge t_2 \ge \cdots \ge t_k \ge 1$. Then

$$\prod_{j=1}^{k} \sum_{i=1}^{t_j} a_{ij} \ge \prod_{j=1}^{k} \sum_{i=1}^{t_j} a^*_{ij}. \tag{1.6}$$

Theorem 3 Let $t_1 \ge t_2 \ge \cdots \ge t_k \ge 1$. If $a_{ij} \le 1$($1 \le i \le t_j$; $1 \le j \le k$), then

$$\prod_{j=1}^{k} (1 + a_{1j} a_{2j} \cdots a_{t_j j}) \le \prod_{j=1}^{k} (1 + a'_{1j} a'_{2j} \cdots a'_{t_j j}). \tag{1.7}$$

If $a_{ij} \ge 1$ ($1 \le i \le t_j$; $1 \le j \le k$), then

$$\prod_{j=1}^{k} (1 + a_{1j} a_{2j} \cdots a_{t_j j}) \le \prod_{j=1}^{k} (1 + a^*_{1j} a^*_{2j} \cdots a^*_{t_j j}). \tag{1.8}$$

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If the condition $t_1 \geq t_2 \geq \cdots \geq t_k \geq 1$ in any one of these theorems is changed to $t_k \geq t_{k-1} \geq \cdots \geq t_1 \geq 1$, then the a'_{ij} and the a^*_{ij} should be changed to each other in the conclusion of that theorem. For example, if $t_1 \leq t_2 \leq \cdots \leq t_k$, then the inequalities (1.4), (1.5) and (1.7) become

$$\sum_{j=1}^{k} \prod_{i=1}^{t_j} a_{ij} \leq \sum_{j=1}^{k} \prod_{i=1}^{t_j} a^*_{ij}, \tag{1.9}$$

$$\sum_{j=1}^{k} \prod_{i=1}^{t_j} a_{ij} \leq \sum_{j=1}^{k} \prod_{i=1}^{t_j} a'_{ij} \tag{1.9'}$$

and

$$\prod_{j=1}^{k} (1 + a_{1j} a_{2j} \cdots a_{t_j j}) \leq \prod_{j=1}^{k} (1 + a^*_{1j} a^*_{2j} \cdots a^*_{t_j j}), \tag{1.10}$$

respectively.

Theorem 4 Suppose $t_j \geq l$ $(1 \leq j \leq k)$, where l is a positive integer. Let $q = \left[\frac{n}{l}\right] - 1$. If $a_{ij} \leq 1$ $(1 \leq i \leq t_j;\ 1 \leq j \leq k)$, then

$$\sum_{j=1}^{k} \prod_{i=1}^{t_j} a_{ij} \leq \sum_{s=0}^{q-1} \prod_{v=1}^{l} a^*_{sl+v} + \prod_{v=lq+1}^{n} a^*_v. \tag{1.11}$$

If $a_{ij} \geq 1$ $(1 \leq i \leq t_j,\ 1 \leq j \leq k)$, then

$$\sum_{j=1}^{k} \prod_{i=1}^{t_j} a_{ij} \leq \sum_{s=0}^{q-1} \prod_{v=1}^{l} a'_{sl+v} + \prod_{v=lq+1}^{n} a'_v. \tag{1.12}$$

Theorem 5 Suppose $t_j \geq l$ $(1 \leq j \leq k)$, where l is a positive integer. Let $q = \left[\frac{n}{l}\right] - 1$. f $a_{ij} \leq 1$ $(1 \leq i \leq t_j;\ 1 \leq j \leq k)$, then

$$\prod_{j=1}^{k} (1 + a_{1j} a_{2j} \cdots a_{t_j j}) \leq \prod_{s=0}^{q-1} (1 + a^*_{sl+1} a^*_{sl+2} \cdots a^*_{sl+l})(1 + a^*_{lq+1} a^*_{lq+2} \cdots a^*_n). \tag{1.13}$$

If $a_{ij} \geq 1$ $(1 \leq i \leq t_j,\ 1 \leq j \leq k)$, then

$$\prod_{j=1}^{k} (1 + a_{1j} a_{2j} \cdots a_{t_j j}) \leq \prod_{s=0}^{q-1} (1 + a'_{sl+1} a'_{sl+2} \cdots a'_{sl+l})(1 + a'_{lq+1} a'_{lq+2} \cdots a'_n). \tag{1.14}$$

The proof of Theorems 1 and 2 in [2] relyed on some inequalities due to Ruderman[3].

One of the purposes of this paper is to establish another kind of arrangement inequalities which are in the following theorems.

Theorem 6 Let $t_1 \geq t_2 \geq \cdots t_k \geq 1$. If $a_{ij} \leq 1$ $(1 \leq i \leq t_j,\ 1 \leq j \leq k)$, then

$$\prod_{j=1}^{k} (1 - a_{1j} a_{2j} \cdots a_{t_j j}) \geq \prod_{j=1}^{k} (1 - a'_{1j} a'_{2j} \cdots a'_{t_j j}). \tag{1.15}$$

If $a_{ij} \geq 1$ $(1 \leq i \leq t_j;\ 1 \leq j \leq k)$, then

$$\prod_{j=1}^{k} |1 - a_{1j} a_{2j} \cdots a_{t_j j}| \geq \prod_{j=1}^{k} |1 - a^*_{1j} a^*_{2j} \cdots a^*_{t_j j}| \tag{1.16}$$

If the condition $t_1 \geq t_2 \geq \cdots t_k \geq 1$ in Theorem 6 is changed to $t_k \geq t_{k-1} \geq \cdots \geq t_1 \geq 1$, then (1.15) should be changed to

$$\prod_{j=1}^{k}(1-a_{1j}a_{2j}\cdots a_{t_j j})\geq\prod_{j=1}^{k}(1-a^*_{1j}a^*_{2j}\cdots a^*_{t_j j}). \qquad (1.15')$$

Theorem 7 Suppose $t_j\geq l$ $(1\leq j\leq k)$, where l is a positive integer. Let $q=\left[\frac{n}{l}\right]-1$. If $a_{ij}\leq 1$ $(1\leq i\leq t_j,\ 1\leq j\leq k)$, then

$$\prod_{j=1}^{k}(1-a_{1j}a_{2j}\cdots a_{t_j j})\geq\prod_{s=0}^{q-1}(1-a^*_{sl+1}a^*_{sl+2}\cdots a^*_{sl+l})(1-a^*_{sq+1}a^*_{sq+2}\cdots a^*_n). \qquad (1.17)$$

If $a_{ij}\geq 1$ $(1\leq i\leq t_j,\ 1\leq j\leq k)$, then

$$\prod_{j=1}^{k}|1-a_{1j}a_{2j}\cdots a_{t_j j}|\geq\prod_{s=0}^{q-1}|1-a'_{sl+1}a'_{sl+2}\cdots a'_{sl+l}|\cdot|1-a'_{lq+1}a'_{lq+2}\cdots a'_n|. \qquad (1.18)$$

The other purpose of this paper is to provide a simple proof which is an unified treatment for all the inequalities in Theorems 1-7 except the inequality (1.12), and the treatment does not rely on the Ruderman's inequalities.

We also point out here that the inequality (1.12) is not correct. When $a_{ij}\geq 1$, the inequality (17) in [2] is not true, so the proof of (10) there (our (12)) is incorrect. This also can be seen from the following counterexample:

Let $k=2$, $t_1=3$, $t_2=4$, $l=2$, and

$$(a_{11},a_{21},a_{31})=(1,1,2),$$
$$(a_{12},a_{22},a_{32},a_{42})=(3,3,4,5).$$

Then the value of the right-hand side of (1.12) in this case is

$$1\cdot 1+2\cdot 3+3\cdot 4\cdot 5=67;$$

but the value of the left-hand side of (1.12) is

$$1\cdot 1\cdot 2+3\cdot 3\cdot 4\cdot 5=182.$$

According to (1.12), we would have

$$182\leq 67,$$

which is impossible.

II. Lemmas

We first prove some lemmas.

Lemma 1 Let $(c_1,c_2,\cdots,c_p)$ and $(c_{p+1},c_{p+2},\cdots,c_{p+q})$ be two non-negative sequences of length p and length q, respectively. Let $c'_1,c'_2,\cdots,c'_{p+q}$ be the c_i $(1\leq i\leq p+q)$ arranged in non-decreasing order: $c'_1\leq c'_2\leq\cdots\leq c'_{p+q}$, and $c^*_1,c^*_2,\cdots,c^*_{p+q}$ be the c_i $(1\leq i\leq p+q)$ arranged in non-increasing order: $c^*_1\geq c^*_2\geq\cdots\geq c^*_{p+q}$. Let $p\geq q\geq l$, where l is a positive integer. Then

$$(c_1+c_2+\cdots+c_p)(c_{p+1}+c_{p+2}+\cdots+c_{p+q})\geq(c^*_1+c^*_2+\cdots+c^*_p)(c^*_{p+1}+c^*_{p+2}+\cdots+c^*_{p+q}) \qquad (2.1)$$

If $c_i\leq 1$ $(1\leq i\leq p+q)$, then

$$(1+c_1c_2\cdots c_p)(1+c_{p+1}c_{p+2}\cdots c_{p+q})\leq(1+c'_1c'_2\cdots c'_p)(1+c'_{p+1}c'_{p+2}\cdots c'_{p+q}), \qquad (2.2)$$

$$(1-c_1c_2\cdots c_p)(1-c_{p+1}c_{p+2}\cdots c_{p+q})\geq(1-c'_1c'_2\cdots c'_p)(1-c'_{p+1}c'_{p+2}\cdots c'_{p+q}), \qquad (2.3)$$

$$c_1c_2\cdots c_p+c_{p+1}c_{p+2}\cdots c_{p+q}\leq c'_1c'_2\cdots c'_p+c'_{p+1}c'_{p+2}\cdots c'_{p+q}, \qquad (2.4)$$

$$c_1c_2\cdots c_p+c_{p+1}c_{p+2}\cdots c_{p+q}\leq c^*_1c^*_2\cdots c^*_l+c^*_{l+1}c^*_{l+2}\cdots c^*_{p+q}. \qquad (2.5)$$

And if $c_i \geq 1$ $(1 \leq i \leq p+q)$, then

$$c_1 c_2 \cdots c_p + c_{p+1} c_{p+2} \cdots c_{p+q} \leq c'_1 c'_2 \cdots c'_l + c'_{l+1} c'_{l+2} \cdots c'_{p+q} \tag{2.6}$$

$$c_1 c_2 \cdots c_p + c_{p+1} c_{p+2} \cdots c_{p+q} \leq c^*_1 c^*_2 \cdots c^*_p + c^*_{p+1} c^*_{p+2} \cdots c^*_{p+q}. \tag{2.7}$$

Proof Without loss of generality, we suppose that all c_i are positive. We introduce the following notations:

$$\mathrm{S}_1 = \{c_1, c_2, \cdots, c_p\}, \quad \mathrm{S}_2 = \{c_{p+1}, c_{p+2}, \cdots, c_{p+q}\},$$
$$\mathrm{S}^*_1 = \{c^*_1, c^*_2, \cdots, c^*_p\}, \quad \mathrm{S}^*_2 = \{c^*_{p+1}, c^*_{p+2}, \cdots, c^*_{p+q}\},$$
$$\mathrm{S}'_1 = \{c'_1, c'_2, \cdots, c'_p\}, \quad \mathrm{S}'_2 = \{c'_{p+1}, c'_{p+2}, \cdots, c'_{p+q}\},$$
$$\mathrm{S}^*_3 = \{c^*_1, c^*_2, \cdots, c^*_l\}, \quad \mathrm{S}^*_4 = \{c^*_{l+1}, c^*_{l+2}, \cdots, c^*_{p+q}\},$$
$$\mathrm{S}'_3 = \{c'_1, c'_2, \cdots, c'_l\}, \quad \mathrm{S}'_4 = \{c'_{l+1}, c'_{l+2}, \cdots, c'_{p+q}\},$$

and

d_1: sum of the elements in $\mathrm{S}_1 \cap \mathrm{S}^*_1$, e_1: sum of the elements in $\mathrm{S}_2 \cap \mathrm{S}^*_1$,
d_2: product of the elements in $\mathrm{S}_1 \cap \mathrm{S}'_1$, e_2: product of the elements in $\mathrm{S}_2 \cap \mathrm{S}'_1$,
d_3: product of the elements in $\mathrm{S}_1 \cap \mathrm{S}^*_3$, e_3: product of the elements in $\mathrm{S}_2 \cap \mathrm{S}^*_3$,
d_4: product of the elements in $\mathrm{S}_1 \cap \mathrm{S}'_3$, e_4: product of the elements in $\mathrm{S}_2 \cap \mathrm{S}'_3$,
f_1: sum of the elements in $\mathrm{S}_1 \cap \mathrm{S}^*_2$, g_1: sum of the elements in $\mathrm{S}_2 \cap \mathrm{S}^*_2$,
f_2: product of the elements in $\mathrm{S}_1 \cap \mathrm{S}'_2$, g_2: product of the elements in $\mathrm{S}_2 \cap \mathrm{S}'_2$,
f_3: product of the elements in $\mathrm{S}_1 \cap \mathrm{S}^*_4$, g_3: product of the elements in $\mathrm{S}_2 \cap \mathrm{S}^*_4$,
f_4: product of the elements in $\mathrm{S}_1 \cap \mathrm{S}'_4$, g_4: product of the elements in $\mathrm{S}_2 \cap \mathrm{S}'_4$.

Since

$$d_1 + f_1 = c_1 + c_2 + \cdots + c_p, \quad e_1 + g_1 = c_{p+1} + c_{p+2} + \cdots + c_{p+q},$$
$$d_1 + e_1 = c^*_1 + c^*_2 + \cdots + c^*_p, \quad f_1 + g_1 = c^*_{p+1} + c^*_{p+2} + \cdots + c^*_{p+q},$$
$$c_1 + c_2 + \cdots + c_p \leq c^*_1 + c^*_2 + \cdots + c^*_p,$$
$$c_{p+1} + c_{p+2} + \cdots + c_{p+q} \leq c^*_1 + c^*_2 + \cdots + c^*_q \leq c^*_1 + c^*_2 + \cdots + c^*_p,$$

we have

$$f_1 \leq e_1 \quad \text{and} \quad g_1 \leq d_1.$$

Hence $(e_1 - f_1)(d_1 - g_1) \geq 0$, i.e., $d_1 e_1 + f_1 g_1 \geq d_1 f_1 + e_1 g_1$. Therefore,

$$(d_1 + f_1)(e_1 + g_1) \geq (d_1 + e_1)(f_1 + g_1).$$

This is (2.1).

Now suppose $c_j \leq 1$ $(1 \leq j \leq p+q)$. Since

$$d_2 f_2 = c_1 c_2 \cdots c_p, \quad e_2 g_2 = c_{p+1} c_{p+2} \cdots c_{p+q},$$
$$d_2 e_2 = c'_1 c'_2 \cdots c'_p, \quad f_2 g_2 = c'_{p+1} c'_{p+2} \cdots c'_{p+q},$$
$$c_{p+1} c_{p+2} \cdots c_{p+q} \leq c'_{p+1} c'_{p+2} \cdots c'_{p+q},$$
$$c_1 c_2 \cdots c_p \leq c'_{q+1} c'_{q+2} \cdots c'_{p+q} \leq c'_{p+1} c'_{p+2} \cdots c'_{p+q},$$

we have $e_2 \leq f_2$ and $d_2 \leq g_2$. Hence $(e_2 - f_2)(d_2 - g_2) \geq 0$, i.e.

$$d_2 e_2 + f_2 g_2 \geq d_2 f_2 + e_2 g_2. \tag{2.8}$$

This is (2.4). From (2.8), we have

$$(1 + d_2 e_2)(1 + f_2 g_2) \geq (1 + d_2 f_2)(1 + e_2 g_2),$$

$$(1-d_2e_2)(1-f_2g_2)\leq(1-d_2f_2)(1-e_2g_2).$$

These are (2.2) and (2.3), respectively.

Since

$$d_3f_3=c_1c_2\cdots c_p,\quad e_3g_3=c_{p+1}c_{p+2}\cdots c_{p+q},$$
$$d_3e_3=c_1^*c_2^*\cdots c_l^*,\quad f_3g_3=c_{l+1}^*c_{l+2}^*\cdots c_{p+q}^*,$$
$$c_1c_2\cdots c_p\leq c_1^*c_2^*\cdots c_p^*\leq c_1^*c_2^*\cdots c_l^*,$$
$$c_{p+1}c_{p+2}\cdots c_{p+q}\leq c_1^*c_2^*\cdots c_q^*\leq c_1^*c_2^*\cdots c_l^*,$$

we have $e_3\geq f_3$ and $d_3\geq g_3$. Hence $(e_3-f_3)(d_3-g_3)\geq 0$, i.e.

$$d_3e_3+f_3g_3\geq d_3f_3+e_3g_3.$$

This is (2.5).

Now suppose $c_j\geq 1$ $(1\leq j\leq p+q)$.

Since

$$d_4f_4=c_1c_2\cdots c_p,\quad e_4g_4=c_{p+1}c_{p+2}\cdots c_{p+q},$$
$$d_4e_4=c_1'c_2'\cdots c_l',\quad f_4g_4=c_{l+1}'c_{l+2}'\cdots c_{p+q}',$$
$$c_1c_2\cdots c_p\leq c_{q+1}'c_{q+2}'\cdots c_{q+q}'\leq c_{l+1}'c_{l+2}'\cdots c_{p+q}',$$
$$c_{p+1}c_{p+2}\cdots c_{p+q}\leq c_{p+1}'c_{p+2}'\cdots c_{p+q}'\leq c_{l+1}'c_{l+2}'\cdots c_{p+q}',$$

We have $g_4\geq d_4$ and $f_4\geq e_4$. Hence $(e_4-f_4)(d_4-g_4)\geq 0$, i.e.

$$d_4e_4+f_4g_4\geq d_4f_4+e_4g_4.$$

This is (2.6).

Since

$$d_1f_1=c_1c_2\cdots c_p,\quad e_1g_1=c_{p+1}c_{p+2}\cdots c_{p+q},$$
$$d_1e_1=c_1^*c_2^*\cdots c_p^*,\quad f_1g_1=c_{p+1}^*c_{p+2}^*\cdots c_{p+q}^*,$$
$$c_1c_2\cdots c_p\leq c_1^*c_2^*\cdots c_p^*,$$
$$c_{p+1}c_{p+2}\cdots c_{p+q}\leq c_1^*c_2^*\cdots c_q^*\leq c_1^*c_2^*\cdots c_p^*,$$

we have $e_1\geq f_1$ and $d_1\geq g_1$. Hence $(e_1-f_1)(d_1-g_1)\geq 0$ i.e., $d_1f_1+e_1g_1\leq d_1e_1+f_1g_1$. This is (2.7). ∎

Lemma 2 Let $(c_1, c_2, \cdots, c_p)$ and $(c_{p+1}, c_{p+2}, \cdots, c_{p+q})$ be two nonnegative sequences of length p and length q, respectively, and $p, q\geq l$, where l is a positive integer. If $c_i\leq 1$ $(1\leq i\leq p+q)$ and the sequence $(c_1, \cdots, c_p, c_{p+1}, \cdots, c_{p+q})$ is non-increasing, or $c_i\geq 1$ $(1\leq i\leq p+q)$ and the sequence $(c_1, \cdots, c_p, c_{p+1}, \cdots, c_{p+q})$ is non-decreasing, then

$$c_1c_2\cdots c_p+c_{p+1}c_{p+2}\cdots c_{p+q}\leq c_1c_2\cdots c_l+c_{l+1}c_{l+2}\cdots c_{p+q},\tag{2.9}$$
$$(1+c_1c_2\cdots c_p)(1+c_{p+1}c_{p+2}\cdots c_{p+q})\leq(1+c_1c_2\cdots c_l)(1+c_{l+1}c_{l+2}\cdots c_{p+q}),\tag{2.10}$$
$$(1-c_1c_2\cdots c_p)(1-c_{p+1}c_{p+2}\cdots c_{p+q})\geq(1-c_1c_2\cdots c_l)(1-c_{l+1}c_{l+2}\cdots c_{p+q}).\tag{2.11}$$

Proof First suppose that $c_i\leq 1$ $(1\leq i\leq p+q)$ and that the sequence $(c_1, \cdots, c_p, c_{p+1}, \cdots, c_{p+q})$ is non-increasing. Sinse

$$1\geq c_{l+1}c_{l+2}\cdots c_p\quad,$$
$$c_1c_2\cdots c_l\geq c_{p+q-l+1}c_{p+q-l+2}\cdots c_{p+q}\geq c_{p+1}c_{p+2}\cdots c_{p+q},$$

we have

$$(1-c_{l+1}c_{l+2}\cdots c_p)(c_1c_2\cdots c_l-c_{p+1}c_{p+2}\cdots c_{p+q})\geq 0. \tag{2.12}$$

This is equivalent to (2.9), and then to any one of (2.10) and (2.11).

Now suppose that $c_i\geq 1$ $(1\leq i\leq p+q)$, and that the sequence $(c_1,c_2,\cdots,c_{p+q})$ is non-decreasing. Since

$$1\leq c_{l+1}c_{l+2}\cdots c_p,\ c_1c_2\cdots c_l\leq c_{p+q-l+1}c_{p-q-l+2}\cdots c_{p+q}\leq c_{p+1}c_{p+2}\cdots c_{p+q},$$

(2.12) and (2.9) still hold. ∎

Lemma 3 Let $0\leq c_i\leq 1$ $(1\leq i\leq p)$ and $p\geq l$, wkere l is a positive integer. Then

$$c_1c_2\cdots c_p\leq c_1c_2\cdots c_l+c_{l+1}c_{l+2}\cdots c_p, \tag{2.13}$$

$$1+c_1c_2\cdots c_p\leq(1+c_1c_2\cdots c_l)(1+c_{l+1}c_{l+2}\cdots c_p), \tag{2.14}$$

$$1-c_1c_2\cdots c_p\geq(1-c_1c_2\cdots c_l)(1-c_{l+1}c_{l+2}\cdots c_p). \tag{2.15}$$

Proof Obvious. ∎

Ⅲ. Proof of theorems

Applying (2.4), (2.7), (2.1) (2.2) and (2.3) repeatedly, we have (1.4), (1.5), (1.6), (1.7) and (1.15), respectively.

Applying (1.9), (2.9) and (2.13) repeatedly, we have (1.11).

Applying (1.10), (2.10) and (2.14) repeatedly, we have (1.13).

Applying (1.15′), (2.11) and (2.15) repeatedly, we have (1.17).

Applying (1.7), (1.13), (1.15) and (1.17) to the $\frac{1}{a_{ij}}$ when $a_{ij}\geq 1$ $(1\leq i\leq t_j,$ $1\leq j\leq k)$, we have (1.8), (1.14), (1.16) and (1.18), respectively.

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