J. Math. Res. &. Exposition, 1987, No.4

# Complete Totally Real Submanifolds with Parallel Mean Curvature*

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(Dedicated to Professor Bai Zheng-Guo on the Occasion of his 70-th Birthday)

## 1. Introduction

A submanifold M in a Kaehler manifold $\overline{M}$ is said to be totally real if every tangent space of M is mapped into its normal space by the complex structure of M. Some fundamental properties of totally real submanifolds can be found in [1], [2]. Let $\sigma$ be the second fundamental form of M. The mean curvature $\eta$ of M is defined by $\eta = \operatorname{tr}\sigma$, and M is called a submanifold with parallel mean curvature if either $\eta = 0$ or $\|\eta\| = \text{constant} \neq 0$ and $\eta/\|\eta\|$ is parallel in the normal bundle over M. M is said to be pseudo-umbilical if it is umbilical with respect to the normal direction of $\eta$.

It is interesting to study totally real submanifolds in the complex number space $C^n$ with parallel meal curvature, and some classifications of such compact totally real submanifolds have been obtained in [1], [2]. In this paper, by employing the generalized maximum principle, we shall prove the following

**Theorem 1** Let M be an $n(\geqslant 2)$-dimensional, non-compact, complete totally real submanifold in $C^n$ with parallel mean curvature. If the second fundamental form $\sigma$ of M satisfies

$$\|\sigma\|^2 \leqslant \|\operatorname{tr}\sigma\|^2/(n-1), \tag{1}$$

then M must be a flat submanifold which is either $R^n$ or a product $S^1 \times R^{n-1}$.

The proof of Theorem 1 is based on the following

**Theorem 2.** Let M be an $n(\geqslant 3)$ dimensional, non-compact, complete totally real submanifold in $C^n$ with nonzero parallel mea curvature. If the inequality (1) holds, then either M is pseudo-umbilical or $\|\sigma'\|^2 = \|\operatorname{tr}\sigma\|^2/(n-1)$, where $\sigma'$ is the second fundamental form of M with respect to the normal direction of $\eta$.

Throughout this paper, all manifolds considered are smooth and connected, and the following ranges of indices will be used:

* Received May 16, 1984. Projects supported by the Science Fund of the Chinese Academy of Sciences.

$A, B, C, \ldots = 1, 2, \cdots, n, 1^*, \cdots, n^*; i, j, k, \cdots = 1, \cdots, n.$

## 2. Preliminaries

Let M be an $n$-dimensional totally real submanifold in the complex $n$-space $C^n$ with the complex structure $J$. we choose a local field of orthonormal frames $\{e_A\}$ in $C^n$ such that, restricted to M, $\{e_i\}$ are tangent to M. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the field of dual frames of $\{e_A\}$ and the connection forms, respectively. Restricting those forms to M, we have (cf. [1] and [3])

$$\omega_{k^*} = 0, \qquad \omega_{k^*i} = \Sigma h^{k^*}_{ij}\omega_j, \qquad h^{k^*}_{ij} = h^{k^*}_{ji}. \tag{2}$$

$$d\omega_{ij} = -\Sigma\omega_{ik}\wedge\omega_{kj} + \frac{1}{2}\Sigma R_{ijkl}\omega_k\wedge\omega_l,$$

$$R_{ijkl} = \Sigma(h^{m^*}_{ik}h^{m^*}_{jl} - h^{m^*}_{il}h^{m^*}_{jk}).$$

$$d\omega_{k^*l^*} = -\Sigma\omega_{k^*j}\wedge\omega_{j^*l^*} + \frac{1}{2}\Sigma R_{k\,l\,ij}\omega_i\wedge\omega_j,$$

$$R_{k^*l^*ij} = \Sigma(h^{k^*}_{im}h^{l^*}_{jm} - h^{l^*}_{im}h^{k^*}_{jm}), \tag{4}$$

$$\sigma = \Sigma h^{k^*}_{ij}\omega_i\otimes\omega_j\otimes e_{k^*}, \tag{5}$$

$$\eta = \mathrm{tr}\,\sigma = \Sigma(\mathrm{tr}\,H^{k^*})e_{k^*}, \qquad H^{k^*} = (h^{k^*}_{ij}). \tag{6}$$

From (3), (5) and (6) the scalar curvature $\rho$ of M is

$$\rho = H^2 - \|\sigma\|^2, \text{ where } H^2 \overset{def.}{=\!=} \|\eta\|^2. \tag{7}$$

If $\eta \neq 0$, we can choose $e_{1^*}$ in such a way that its direction coincides with that of $\eta$. Then

$$\mathrm{tr}H^{1^*} = H, \quad \mathrm{tr}H^{j^*} = 0 \quad (j \neq 1). \tag{8}$$

By virtue of (3), (7) and (8), we have

**Lemma I** Let M be an $n$-dimensional totally real submanifold in $C^n$. If $\rho \geqslant (n-2)\|\sigma\|^2 + 2(n-1)c$ at a point $x \in M$ for some real number $c$, then the sectional curvatures of M at the point $x$ are $\geqslant c$. The proof of this lemma is completely similar to that of [4], and is therefore omitted.

On putting

$$\mu = \sum_{i,j}\Big(h^{1^*}_{ij} - \frac{H}{n}\delta_{ij}\Big)\omega_i\otimes\omega_j\otimes e_{1^*}, \quad \tau = \sum_{\substack{m\neq 1\\ i,j}} h^{m^*}_{ij}\,\omega_i\otimes\omega_j\otimes e_{m^*}, \tag{9}$$

we have

$$\mathrm{tr}\mu = 0, \qquad \|\mu\|^2 = \|\sigma'\|^2 - \frac{H^2}{n}, \qquad \|\sigma'\|^2 = \mathrm{tr}(H^{1^*})^2, \tag{10}$$

$$\mathrm{tr}\,\tau = 0, \qquad \|\tau\|^2 = \sum_{m\neq 1}\mathrm{tr}(H^{m^*})^2,$$

$$\|\sigma\|^2 = \|\mu\|^2 + \|\tau\|^2 + H^2/n, \tag{11}$$

from which it may be seen that $\|\tau\|^2$ as well as $\|\mu\|^2$ is independent of the choice of the frames and is a globally defined function on M. From now on we choose the local frames $\{e_A\}$ in $C^n$ such that $e_{i^*}=Je_i$ and $e_{1^*}=\eta/\|\eta\|$ if $\eta\neq 0$. Then, by (2) we have (cf. [1])

$$h_{ij}^{k^*}=h_{ji}^{k^*}=h_{kj}^{i^*}=h_{ik}^{j^*} \quad . \tag{12}$$

From (8),(10) and (12) one can easily see the following

**Lemma 2** M is pseudo-umbilical iff $\|\mu\|^2=0$, and M is totally geodesic iff it is pseudo-umbilical and $\|\tau\|^2=0$.

Now assume that $\eta=He_{1^*}$ is parallel, *i.e.*, $H=$constant and $\omega_{1^*k^*}=0$. Using the same calculation as in [3], by means of (3),(4) and (8) we can get

$$\frac{1}{2}\triangle(\|\mu\|^2)=\|D\mu\|^2-[\mathrm{tr}(H^{1^*})^2]^2+H\mathrm{tr}(H^{1^*})^3-\sum_{k\neq 1}[\mathrm{tr}(H^{k^*}H^{1^*})]^2 \ , \tag{13}$$

$$\frac{1}{2}\triangle(\|\tau\|^2)=\|D\tau\|^2+\sum_{i,j\neq 1}\{\mathrm{tr}(H^{i^*}H^{j^*}-H^{j^*}H^{i^*})^2-[\mathrm{tr}(H^{i^*}H^{j^*})]^2\}+\frac{H^2}{n}\|\tau\|^2 \ , \tag{14}$$

where D denotes the generalized covariant differentiation and $\Delta$ the Laplacian.

The following generalized maximum principle which is due to Yau, S.T.-Cheng, S.Y.-Motomiya, M. can be found in [5].

**Lemma 3** Let M be a complete Riemannian manifold with Ricci curvature bounded below, and $f$ a $C^2$-function bounded above on M. Then for any $\varepsilon>0$, there exists a point $x\in M$ such that at $x$

(i) $\sup f-\varepsilon<f(x)$,

(ii) $|\mathrm{grad} f|(x)<\varepsilon$,

(iii) $\Delta f(x)<\varepsilon$.

Furthermore, if $f$ has no maximum, then there exists a sequence of positive numbers $\{\varepsilon_\nu\}$ such that $\varepsilon_\nu\to 0(\nu\to\infty)$, and for all $\nu$, (i) may be replaced by

(i′) $\sup f-\varepsilon_\nu<f(x)<\sup f-\frac{1}{2}\varepsilon_\nu$.

## 3. The Proof of Theorem 2.

For the nonzero mean curvature vector $\eta=He_{1^*}$, we shall start with the formula (13). By Schwarz inequality, it follows from (8) and (10) that

$$\sum_{k\neq 1}[\mathrm{tr}(H^{k^*}H^{1^*})]^2=\sum_{k\neq 1}\Big\{\sum_{i,j}h_{ij}^{k^*}(h_{ij}^{k^*}-\frac{H}{n}\delta_{ij})\Big\}^2\leq\|\mu\|^2\|\tau\|^2 . \tag{15}$$

By repeating the same calculations as Okumura, M. in [6], one can get from (13),(15),(11) and (1) that

$$\frac{1}{2}\Delta(\|\mu\|^2)\geq\|D\mu\|^2+\|\mu\|^2\Big\{\frac{H^2}{n}-\frac{n-2}{\sqrt{n(n-1)}}|H|\cdot\|\mu\|-\|\mu\|^2-\|\tau\|^2\Big\}$$

$$\geq\frac{n-2}{\sqrt{n(n-1)}}|H|\cdot\|\mu\|^2\Big\{\frac{|H|}{\sqrt{n(n-1)}}-\|\mu\|\Big\} \quad . \tag{16}$$

Since (1) implies that

$$\|\mu\|^2 \leq \|\sigma\|^2 - \frac{H}{n} \leq H^2/n(n-1) \tag{17}$$

and, by ( 1 ) and ( 7 ) we deduce $\rho \geq (n-2)\|\sigma\|^2$, and hence, by Lemma 1, the sectional curvatures of M are bounded below from 0, we can apply Lemma 3, and from (16) and (17) conclude that either $\|\mu\|^2 = 0$, *i.e.*, by Lemma 2, M is pseudo-umbilical, or

$$\sup\|\mu\| = |H|/\sqrt{n(n-1)} \quad . \tag{18}$$

If $\|\mu\|^2$ attains its maximum at a point of M, then by using Hopf's well-known maximum principle we see from (16) and (17) that $\|\mu\|^2 = \text{constant}$ and hence $\|\mu\|^2 = H^2/n(n-1)$ on M everywhere, *i.e.*, $\|\sigma'\|^2 = \|\mathrm{tr}\sigma\|^2/(n-1)$ by (10).

Now assume $\|\mu\|^2$ has no maximum on M, we prove that it is impossible. In fact, from Lemma 3 it follows that, for any natural number $\nu$, there exists a point $x_\nu \in M$ such that, by (18) and (16),

$$\frac{H^2}{n(n-1)} - \frac{1}{\nu} < \|\mu\|^2(x_\nu) < \frac{H^2}{n(n-1)} - \frac{1}{2\nu} \tag{19}$$

and

$$\frac{n-2}{\sqrt{n(n-1)}}|H| \cdot \|\mu\|^2(x_\nu)\left\{\frac{H}{\sqrt{n(n-1)}} - \|\mu\|(x_\nu)\right\} < \frac{1}{2\nu} \quad . \tag{20}$$

From (19) we get

$$\frac{1}{2\nu(|H|/\sqrt{n(n-1)} + \|\mu\|(x_\nu))} < \frac{|H|}{\sqrt{n(n-1)}} - \|\mu\|(x_\nu)$$

and thus (20) becomes

$$\frac{n-2}{\sqrt{n(n-1)}}|H| \cdot \|\mu\|^2(x_\nu) < \frac{|H|}{\sqrt{n(n-1)}} + \|\mu\|(x_\nu)$$

or

$$\|\mu\|^2(x_\nu) - \frac{\sqrt{n(n-1)}}{(n-2)|H|}\|\mu\|(x_\nu) - \frac{1}{n-2} < 0 \, . \tag{21}$$

Since $\|\mu\|(x_\nu) > 0$, (21) yields

$$\|\mu\|(x_\nu) < (\sqrt{n(n-1)} + \sqrt{n(n-1) + 4(n-2)H^2})/2(n-2)|H| ,$$

and thus

$$\sup\|\mu\| \leq (\sqrt{n(n-1)} + \sqrt{n(n-1) + 4(n-2)H^2})/2(n-2)|H| \, ,$$

from which together with (18) it follows that

$$2(n-2)H^2 - n(n-1) \leq \sqrt{n^2(n-1)^2 + 4n(n-1)(n-2)H^2} \quad .$$

In view of $H \neq 0$, we have then

$$H^2 \leq 2n(n-1)/(n-2) \, . \tag{22}$$

We now consider a homothetic transformation $\pi$ in $C^n$ which is defined by $\omega_A = \lambda\omega_A$, where $\lambda$ is an arbitrary positive real number. Then, by the structure equations, we have $\bar{\omega}_{AB}=\omega_{AB}$. Thus, it is easy to see that the image $\bar{M}=\mathscr{H}(M)$ satisfies the same conditions as M and $\bar{H}^2=H^2/\lambda^2$, where $\bar{H}$ is the corresponding quantity for $\bar{M}$. Then we must have, as (22) above,

$$H^2=\lambda^2\bar{H}^2<2\lambda^2 n(n-1)/(n-2),$$

which is evidently absurd for $\lambda<\sqrt{(n-2)H^2/2(n-1)n}$. This completes the proof of Theorem 2.

## 4. The Proof of Theorem 1.

First of all, we note that $\|\mathrm{tr}\sigma\|^2=H^2=$ comstant under the hypothesis of Theorem 1. Thus, if $\eta=0$, then (1) implies $\|\sigma\|^2=0$ on M everywhere, *i.e.*, M is a totally geodesic $R^n$ in $C^n$. So from now on we assume $\eta\neq 0$ on M.

If $n>3$, by Theorem 2 we have to consider two cases.

Case (Ⅰ): M is pseudo-umbilical, *i.e.*, $\|\mu\|^2=0$ everywhere. Using the following well-known estimate (cf. [3], § 7)

$$\sum_{i,j\neq 1}\{\mathrm{tr}(H^{i^*}H^{j^*}-H^{j^*}H^{i^*})^2-[\mathrm{tr}(H^{i^*}H^{j^*})]^2\}\geq-(2-\frac{1}{n-1})\|\tau\|^4,$$

we have from (14)

$$\frac{1}{2}\Delta(\|\tau\|^2)\geq(2-\frac{1}{n-1})\|\tau\|^2\left\{\frac{n-1}{n(2n-3)}H^2-\|\tau\|^2\right\}\quad. \tag{23}$$

The condition (1) implies that $\|\tau\|^2(\leq\|\sigma\|^2)$ is bounded above and the sectional curvatures of M are bounded below from 0 (Lemma 1). Applying the assertion (iii) of Lemma 3, (23) gives rise to either $\|\tau\|^2=0$ or

$$\sup\|\tau\|^2\geq(n-1)H^2/n(2n-3)\ . \tag{24}$$

However, by virtue of (11) and the fact that $\|\mu\|^2=0$, it follows from (1) that $\|\tau\|^2<H^2/n(n-1)$, which contradicts (24) for $n>3$. Hence, $\|\tau\|^2=0$ and, by Lemma 2, M is a totally geodesic $R^n$ in $C^n$.

Case (Ⅱ): $\|\sigma'\|^2=\|\mathrm{tr}\sigma\|^2/(n-1)$, then by (10), $\|\mu\|^2=H^2/n(n-1)$ everywhere. In this case, from (11) and (1) we get

$$\|\tau\|^2+H^2/n(n-1)=\|\sigma\|^2-H^2/n<H^2/n(n-1)\ ,$$

which implies $\|\tau\|^2=0$ on M everywhere. Thus, $h_{ij}^{k^*}=0$ $(k\neq 1)$ and, by (12), $h_{jk}^{1^*}=0$ except $j=k=1$, from which it follows that

$$h_{11}^{1^*}=\mathrm{tr}(H^{1^*})=H=\mathrm{constant}\ .$$

If $h_{11}^{1^*}=0$, M is totally geodesic. Since $\|\sigma\|^2=H^2/(n-1)$, by an analogy to the proof of Theorem 4 in [2], $M=S^1\times R^{n-1}$, where $S^1$ is a circle with the radius $1/|H|$ in $C^1$.

Finally, we consider the case that $n=2$. By Lemma 1, the condition (1) guarantees that the Gauss curvature of M is nonnegative. If M is viewed as a surface in $R^4$ with parallel mean curvature, then Hoffman's theorem (cf. [7])

says that M is either $R^2$ or $S^1 \times R^1$ because M is non-compact. Therefore, Theorem 1 is proved completely.

## Referencos

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