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The Tensor Products of Nilrings with Bounded Index of Nilpotence*

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R is a ring, for any nilpotent element $r \in R$ if there exists a fixed integer n such that $r^n = 0$, then R is said to be with bounded index of nilpotence, the least of such integer n is called the index of R, denoted by $i(R)$.

If R is a nil ring with bounded index $i(R)=n$, R' is a commutative ring, A·A·Klein [1] has discussed the property of bounded index of nilpotence of $R \otimes_Z R'$. In this paper we shall discuss the properties of bounhed index of nilpotence of $R \otimes R'$ when R' is not commutative.

First, let us observe the polynomial identities which R satisfies. Obviously

$$g(x) = X^n \qquad (1)$$

is a polynomial identity of R, because of R being nil with $i(R)=n$. Multilinearizing $g(X)$ [2, p.6], we obtain that

$$f(X_1, X_2, \cdots, X_n) = \sum_{\pi \in \mathrm{sym}(n)} X_{\pi(1)} X_{\pi(2)} \cdots X_{\pi(n)}, \qquad (2)$$

where π takes over the symmetric group $\mathrm{sym}(n)$ on n symbols $\{1, 2, \cdots, n\}$, is a polynomial identity of R.

Further, we have:

Lemma 1 Let

$$f_1(X_1, X_2, \cdots, X_{2n}) = \sum_{\substack{\sigma \text{ takes over all} \\ \text{odd permutation} \\ \text{in sym}(2n)}} X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(2n)}, \qquad (3)$$

$$f_2(X_1, X_2, \cdots, X_{2n}) = \sum_{\substack{\tau \text{ takes over all} \\ \text{even permutation} \\ \text{in sym}(2n)}} X_{\tau(1)} X_{\tau(2)} \cdots X_{\tau(2n)}, \qquad (4)$$

then $f_1(X_1, X_2, \cdots, X_{2n})$ and $f_2(X_1, X_2, \cdots, X_{2n})$ are polynomial identities of R.

Proof If $X_{1'} X_{2'} \cdots X_{(2n)'}$ is a summand of $f_1(X_1, X_2, \cdots, X_{2n})$, where $(1', 2', \cdots, (2n)')$, by definition of (3), is on odd permutation of $(1, 2, \cdots, 2n)$, consider

$$f(X_{1'} X_{2'}, X_{3'} X_{4'}, \cdots, X_{(2n-1)'} X_{(2n)'}) . \qquad (5)$$

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Referring to (2), each summand of (5) can be transformed to $X_{1'}X_{2'}\cdots\cdots X_{(2n)'}$ by performing even number transpositions, so each summand of (5) has same odd permutation of $\{1, 2, \cdots, 2n\}$ as its foot index, hence each summand of (5) is also a summand of (3). Moreover, if $X_{1''}X_{2''}\cdots\cdots X_{(2n)''}$ is another summand of (3), but is not a summand of (5), as above, $f(X_{1''}X_{2''}, X_{3''}X_{4''}, \cdots\cdots, X_{(2n-1)''}X_{(2n)''})$ is a partial sum of (3), and (5) and $f(X_{1''}X_{2''}, X_{3''}X_{4''}, \cdots\cdots X_{(2n-1)''}X_{(2n)''})$ have distinct summands. Hence we conclude that $f_1(X_1, X_2, \cdots, X_{2n})$ is a sum of polynomials of the form (5). This implies $f_1(X_1, X_2, \cdots, X_{2n})$ is a polynomial identity of R because of (2) or (5) being polynomial identity of R.

Similar argument is applied to show that (4) is a polynomial identity of R.

Corollary 2 The standard polynomial [2, p13]

$$S_{2n}(X_1, X_2, \cdots\cdots, X_{2n}) = \sum_{\pi\in \operatorname{sym}(2n)} (\operatorname{sg}\pi) X_{\pi(1)}X_{\pi(2)}\cdots\cdots X_{\pi(2n)}$$

is a polynomial identity of R, where $\operatorname{sg}\pi = -1$ if π is an odd permutation, $\operatorname{sg}\pi = 1$ if π is an even permutation.

Proof Since $S_{2n}(X_1, X_2, \cdots\cdots, X_{2n}) = -f_1(X_1, \cdots, X_{2n}) + f_2(X_1, \cdots, X_{2n})$, the result holds obviously by Lemma 1.

Now we transfer to the main topics.

Let R be nil with bounded index $i(\mathrm{R}) = 2$, for each pair $a, b \in \mathrm{R}$, $0 = (a+b)^2 = a^2 + ab + ba + b^2 = ab + ba$, so

$$ab = -ba, \quad a^2 = 0 \quad \text{for any } a, b \in \mathrm{R}.$$

We have:

Theorem 3 If R is nil with bounded index (of nilpotence) $i(\mathrm{R}) = 2$, then for any ring R', $\mathrm{R}\otimes_{\mathbf{Z}}\mathrm{R}'$ is nil.

Further, if R' satisfies any of the following

(i) R' is finitely generated as $\mathbf{Z}$-module with n gererators $x_1, x_2, \cdots, x_n$;

(ii) R' is nil with bounded index of nilpotence $i(\mathrm{R}') = n < \infty$, then $\mathrm{R}\otimes_{\mathbf{Z}}\mathrm{R}'$ is nil with bounded index, and in the first case $i(\mathrm{R}\otimes\mathrm{R}') \leq n+1$, in the second case $i(\mathrm{R}\otimes_{\mathbf{Z}}\mathrm{R}') \leq 2n$.

Proof For any $\sum_{i=1}^{k} a_i \otimes x_i \in \mathrm{R}\otimes\mathrm{R}'$, each summand of the expansion of $(\sum_{i=1}^{k} a_i \otimes x_i)^{k+1}$ is of the form $a_{j_1}a_{j_2}\cdots\cdots a_{j_{k+1}} \otimes x_{j_1}x_{j_2}\cdots\cdots x_{j_{k+1}}$, where $a_{j_i} \in \{a_1, a_2, \cdots, a_k\} \subseteq \mathrm{R}$, $x_{j_i} \in \{x_1, x_2, \cdots, x_k\} \subseteq \mathrm{R}'$, $1 \leq i \leq k+1$. By pigeon-hole principle, some a_i must occur twice in $a_{j_1}a_{j_2}\cdots a_{j_{k+1}}$, so by (6), $a_{j_1}a_{j_2}\cdots a_{j_{k+1}} = 0$, this shows that

$(\sum_{i=1}^{k} a_i \otimes x_i)^{k+1} = 0$, the first assertion is proved.

Now if R′ satisfies (i), $\forall x \in R \otimes R'$, we can write $x = \sum_{i=1}^{n} a_i \otimes m_i x_i$, $a_i \in R$, $m_i \in \mathbf{Z}$, x has at most n summands of the form $a \otimes x_i$, by the above proof, $x^{n+1} = 0$, hence $R \otimes R'$ is nil with bounded index $i(R \otimes R') \leq n+1$.

If R′ satisfies (ii), $\forall x = \sum_{i=1}^{u} a_i \otimes x_i \in R \otimes R'$, $a_i \in R$, $x_i \in R'$, by the linearity of tensor product, we may assume that $a_i's$ are distinct.

Consider $x^{2n} = (\sum_{i=1}^{u} a_i \otimes x_i)^{2n}$, if $u < 2n$, by the above proof, we see $x^{2n} = 0$; if $u \geq 2n$, the expansion of X^{2n} is a sum of the monomials of the form: $a_{t_1} a_{t_2} \cdots\cdots a_{t_{2n}} \otimes x_{t_1} x_{t_2} \cdots\cdots x_{t_{2n}}$, where $a_{t_i} \in \{a_1, a_2, \cdots, a_u\}$, $x_{t_i} \in \{x_1, x_2, \cdots, x_u\}$, if some a_i occurs twice in $a_{t_1} a_{t_2} \cdots a_{t_{2n}}$, then by (6), $a_{t_1} a_{t_2} \cdots a_{t_{2n}} \otimes x_{t_1} x_{t_2} \cdots x_{t_{2n}} = 0$. Deleting all zero summands in the expansion of x^{2n}, we obtain:

$$x^{2n} = (\sum_{i=1}^{u} a_i \otimes x_i)^{2n} = \sum_{\substack{1 \leq t_1, t_2, \cdots, t_{2n} \leq u \\ t_i \text{ are distinct}}} a_{t_1} a_{t_2} \cdots a_{t_{2n}} \otimes x_{t_1} x_{t_2} \cdots x_{t_{2n}}, \qquad (7)$$

if $a_{t_1} a_{t_2} \cdots\cdots a_{t_{2n}} \otimes x_{t_1} x_{t_2} \cdots\cdots x_{t_{2n}}$ is a summand of (7), and τ is any permutation of $\{1, 2, \cdots, 2n\}$ then $a_{t_{\tau(1)}} a_{t_{\tau(2)}} \cdots\cdots a_{t_{\tau(2n)}} \otimes x_{t_{\tau(1)}} x_{t_{\tau(2)}} \cdots\cdots x_{t_{\tau(2n)}}$ is also a summand of (7), so

$$\sum_{\tau \in \mathrm{sym}(2n)} a_{t_{\tau(1)}} a_{t_{\tau(2)}} \cdots\cdots a_{t_{\tau(2n)}} \otimes x_{t_{\tau(1)}} x_{t_{\tau(2)}} \cdots\cdots x_{t_{\tau(2n)}} \qquad (8)$$

isa partial sum of (7). Moreover if $a_{t'_1} a_{t'_2} \cdots\cdots a_{t'_{2n}} \otimes x_{t'_1} x_{t'_2} \cdots\cdots x_{t'_{2n}}$ is a summand of (7) and $\{a_{t_1}, a_{t_2}, \cdots, a_{t_{2n}}\} \neq \{a_{t'_1}, a_{t'_2}, \cdots, a_{t'_{2n}}\}$, then

$$\sum_{\tau \in \mathrm{sym}(2n)} a_{t'_{\tau(1)}} a_{t'_{\tau(2)}} \cdots\cdots a_{t'_{\tau(2n)}} \otimes x_{t'_{\tau(1)}} x_{t'_{\tau(2)}} \cdots\cdots x_{t'_{\tau(2n)}} \qquad (9)$$

is another partial sum of (7), and, (8) and (9) have distinct summands. By these analysis, we have seen that x^{2n} is a sum of elements of the form (8), if (8) vanishes, then so does x^{2n}.

Hence it remains to show that (8) vanishes.

If $\pi \in \mathrm{sym}(2n)$ is an odd (or even) permutation, then by performing odd (respectively even) number times transposition on $a_{t_{\pi(1)}} a_{t_{\pi(2)}} \cdots\cdots a_{t_{\pi(2n)}}$ we can transform $a_{t_{\pi(1)}} a_{t_{\pi(2)}} \cdots\cdots a_{t_{\pi(2n)}}$ to $a_{t_1} a_{t_2} \cdots\cdots a_{t_{2n}}$, by (6) each transposition changes the positive negative sign, so

$$a_{t_{\pi(1)}} a_{t_{\pi(2)}} \cdots\cdots a_{t_{\pi(2n)}} = (\mathrm{sg}\pi) a_{t_1} a_{t_2} \cdots\cdots a_{t_{2n}}.$$

Hence

$$a_{t_{\pi(1)}} a_{t_{\pi(2)}} \cdots\cdots a_{t_{\pi(2n)}} \otimes x_{t_{\pi(1)}} x_{t_{\pi(2)}} \cdots\cdots x_{t_{\pi(2n)}}$$

$$= (\mathrm{sg}\pi) a_{t_1} a_{t_2} \cdots\cdots a_{t_{2n}} \otimes x_{t_{\pi(1)}} x_{t_{\pi(2)}} \cdots\cdots x_{t_{\pi(2n)}}$$
$$= a_{t_1} a_{t_2} \cdots\cdots a_{t_{2n}} \otimes (\mathrm{Sg}\pi) x_{t_{\pi(1)}} x_{t_{\pi(2)}} \cdots\cdots x_{t_{\pi(2n)}} \tag{10}$$

Use (10), rewrite (8) as:

$$x^{2n} = \sum_{\pi \in \mathrm{sym}(2n)} a_{t_{\pi(1)}} a_{t_{\pi(2)}} \cdots\cdots a_{t_{\pi(2n)}} \otimes x_{t_{\pi(1)}} x_{t_{\pi(2)}} \cdots\cdots X_{t_{\pi(2n)}}$$
$$= \sum_{\pi \in \mathrm{sym}(2n)} a_{t_1} a_{t_2} \cdots\cdots a_{t_{2n}} \otimes (\mathrm{sg}\pi) x_{t_{\pi(1)}} x_{t_{\pi(2)}} \cdots\cdots x_{t_{\pi(2n)}}$$
$$= a_{t_1} a_{t_2} \cdots\cdots a_{t_{2n}} \otimes \sum_{\pi \in \mathrm{sym}(2n)} (\mathrm{sg}\pi) x_{t_{\pi(1)}} x_{t_{\pi(2)}} \cdots\cdots x_{t_{\pi(2n)}}$$
$$= a_{t_1} a_{t_2} \cdots\cdots a_{t_{2n}} \otimes S_{2n}(x_{t_1}, x_{t_2}, \cdots\cdots, x_{t_{2n}}) = 0.$$

The last equality holds by corollary 2. (8) vanishes, hence ends our proof.

Partial result of Theorem 3 may be revised slightly. That is:

Corollary 4 If R is nil with bounded index $i(R) = 2$, R' is a ϕ algebra of finite dimension k, then $R \otimes_Z R'$ is nil with bounded index $i(R \otimes R') \leq k + 1$.

Proof For any $x \in R \otimes R'$, x has at most n summands of the form $a \otimes \alpha x$ $a \in R$, $\alpha \in \phi$, $x \in R'$. The corollary holds immediately by Theorem 3.

From now on we assume R is nil with bounded index $i(R) = n \geq 3$. Since R is a nil PI-algebra, so R is locally nilpotent [3. p232]. For any ring R', $x \in R \otimes_Z R'$ has the form $x = \sum_{i=1}^{m} a_i \otimes x_i$, $a_i \in R$, $x_i \in R'$, and $\{a_1, a_2, \cdots, a_m\}$ is a finite subset of R and is then nilpotent, so x is nilpotent, This is:

Theorem 5 If R is nil with bounded index $i(R) = n$, for any ring R', $R \otimes_Z R'$ is a nil ring.

If R' satisfies some condition, $R \otimes_Z R'$ may have bounded index.

Remark Klein has pointed out that if R is nil with $i(R) = n < \infty$, then $M_k(R)$ is nil with bounded index [1]. The author has proved this result independently and given a bound for $i(M_k(R))$. Using this result we establish:

Theorem 6 If R is nil with bounded index $i(R) = n < \infty$, R' is an algebra over a field ϕ of finite dimension m with unit 1, then $R \otimes R'$ is nil with bounded index of nilpotence.

Proof First consider $M_m(\phi) \otimes_Z R \cong \phi \otimes_Z M_m(R)$. By the above remark, $M_m(R)$ has bounded index, field ϕ is commutative, by the result which A·A. Klein proved in [1], $\phi \otimes_Z M_m(R)$ has bounded index of nilpotence, so does $M_m(\phi) \otimes_Z R$. Now using the usual method, for any $x \in R'$, let x_L denote the linear transformation of R' over ϕ such that for $a \in R'$, $x_L(a) = xa$. Since $1 \in R'$, so R' is isomorphic to the subalgebra $R'_L = \{x_L \mid x \in R'\}$ of the algebra of linear transfomation of R' over ϕ. Moreover, R' is of finite dimension over ϕ, so R'_L is isomorphic to a subalgebra of $M_m(\phi)$, hence R' is isomorphic to a subalgebra of $M_m(\phi)$. These induce an isomorphic of $R\otimes_Z R'$ onto a subalgebra of $R\otimes_Z M_m(\phi)$, the latter has bounded index of nilpotence, so does the former.

References

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