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Primitive Topological Algebras And Modules*

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In this paper we define a topology that we call canonical on an irreducible module for a topological algebra. Under some additional conditions we prove theorems of the strict density and the uniqueness of the topology for a primitive locally convex F-algebra. We also show that a locally finite primitive locally convex F-algebra is finite dimensional.

All our spaces and algebras are over the scalar field K, which is either the reals or the complexes. By modules we mean algebraic right modules. We follow [1] for notations and terminologies not stated here in general algebra.

By a topological algebra we mean an algebra having a vector topology for which the multiplication is separately continuous i.e. the operations $x \mapsto xy$ and $x \mapsto yx$ are continuous for every y in A. An F-algebra is a complete, metrizable one. A module M for a topological algebra A is called a topological A-module if M is a topological vector space such that

$$(x, a) \mapsto xa, \quad M \times A \to M$$

is continuous for each variable.

Let A be an algebra. An A-module M is irreducible if it has no proper submodules and $MA = \{\sum ma: m \in M, a \in A\} \neq (0)$. M is faithful if $0 \neq a \in A$ implies $Ma \neq (0)$. An algebra is primitive if it has a fait ful irreducible module.

I Now we come to introduce the canonical topology and discuss briefly its elementary properties.

Let A be a topological algebra and M an A-irreducible modume. Thus $M = mA$ for every $0 \neq m \in M$. For a fixed $0 \neq m \in M$, the mapping

$$A \to M: \ a \mapsto ma, \ a \in A$$

is obviously linear and onto, which therefore induces a quotient topology T_m on M. The kernel of this mapping is $(0:m) = \{a \in A: ma = 0\}$. So the quotient space $A/(0:m)$ is topologically isomorphic to the topological vector space $M[T_m]$ It follows that T_m is separated if and only if $(0:m)$ is closed.

It is easy to see that T_m is independent of the choice of $0 \neq m \in M$. This topology is called the canonical topology of the A-module M, and denoted byT.

Lemma 1 M[T] is a topological A-module. Moreover, if $M[T_1]$ is a

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topological A-module too, then $T_1 \subset T$.

Proof Take any $0 \neq m \in M$, we have $M = mA$. For any $a \in A$, $x = mb \in M$ ($b \in A$) and $U \in N(A)$. (We denote by $N(X)$ or $N(\tau)$ the neighbourhood system of zero in a topological vector space $X[\tau]$. Since the mappings $y \mapsto by$ and $y \mapsto ya$ are continuous from A to A, there are V and $W \in N(A)$ such that $bV \subset U$ and $Wa \subset U$. Hence $x(a+V) \subset xa + mU$ and $(x+mW)a \subset xa + mU$. This proves the module operation is separately continuous (to $M[T]$). So $M[T]$ is a topological A-module.

If T_1 is another A-module topology on M. For any $V \in N(T_1)$, since $(x, a) \mapsto xa$ is separately continuous (to $M[T_1]$), there is $U \in N(A)$ such that $mU \subset V$, but $mU \in N(T)$. So $V \in N(T)$, this implies $T_1 \subset T$.

Lemma 2 $M[T]$ is separated if and only if $(0:m)$ is closed for some (hence every) $0 \neq m \in M$.

Let $D = End_A(M)$ denote the set of linear transformations f on M satisfying $f(ma) = f(m))a$ for all $m \in M$ and $a \in A$. Then every f in D is one-to-one and onto by Schur Lemma [1,p171].

Lemma 3 Every $f \in D$ is continuous on $M[T]$.

Proof Take any $0 \neq m \in M$. It is sufficient to show that for any $U \in N(A)$, there is $V \in N(A)$ such that $f(mV) \subset mU$. Assume $fm \neq 0$ (otherwise, $f = 0$), then $(fm)A = M$ and $T = T_{fm}$. Since $mU \in N(T)$, there is $V \in N(A)$ such that $(fm)V \subset mU$. But $(fm)V = f(mV)$.

Lemma 4 Let A be a topological algebra. Suppose $I = eA$ is a minimal right ideal of A, $e \in A$ with $e^2 = e$. Then both I and $(1-e)A$ are closed, and $A = eA \oplus (1-e)A$ (a topologically direct sum).

Proof. Let $x_a \in eA$ and $x_a \to x \in A$, then $ex_a \to ex$. But $ex_a = x_a$, so $ex = \lim ex_a = \lim x_a = x$. It follows $x \in eA$. So eA is closed. That $(1-e)A$ is closed can be proved by same way.

Since $a \mapsto ea$ is continuous, $A = eA \oplus (1-e)A$ is a topologically direct sum.

Theorem 1 Suppose A is a primitive topological algebra with minimal one sided ideals. Then

(1) A has both minimal right ideals and minimal left ideals, all of them are closed;

(2) every minimal right ideal is an A-faithful irreducible module and the canonical topology coincides with its relative topology;

(3) the canonical topology of every A-faithful irreducible module is separated. Moreover, all of A-faithful irreducible modules (with the canonical topology) are topologically (module) isomorphic one another.

Proof. Theorem 7.5.2 of [1] shows A has both minimal right and left ideals, and they have the forms eA and Ae which are also right and left faithful irreducible modules for A, with $e \in A$ and $e^2 = e$. All of them are closed by Lemma 4. This proves (1).

Let τ denote the restriction on eA of the topology of A. Then Lemma 1 shows $\tau \subset T$ (the canonical topology of eA). On the other hand, every $V \in N(T)$ has the form eU with $U \in N(A)$. Note that $U \cap eA \in N(\tau)$ and $U \cap eA = e(U \cap eA) \subset eU$, we have $eU \in N(\tau)$. So $\tau \supset T$ and hence $\tau = T$ and (2) is proved.

Suppose M is any A-faithful irreducible. Since $eA \neq (0)$, there is $m \in M$ such that $meA \neq (0)$. It follows that $meA = M$ since M has no nonzero proper submodules. The mapping $eA \to M$: $ea \to mea$, $a \in A$ is obviously an A-module homomorphism from eA onto M. On the other hand, it is obvious that $(1-e)A \subset (0:me)$ and hence $(1-e)A = (0:me)$ since both of them are regular maximal rigut ideals of A[1,p165]. This shows $mea = 0$ if and only if $ea = 0$. So the mapping is an A-module isomorphism. For any $U \in N(A)$, eU corresponds meU under this isomorphism. It follows that M i topologically isomorphic to eA(each has the canonical topology). The proof of (3) is completed.

II Let A be a locally convex F-algebra and M an A-irreducible module. Then M[T] is a locally convex F-space. We assume that the canonical topology T is separated.

Lemma 5 $D = \mathrm{End}_A(M)$ is isomorphic to either the reals or the complexes or the quaternions. In particular, if A is complex, then D is isomorphic to the complexes.

Proof Take any $0 \neq m \in M$. Let $B = \{b \in A: b(0:m) \subset (0:m)\}$ be the idealizor of $(0:m)$.Since$(0:m)$ is closed,B is a closed subalgebra of A and $(0:m)$ is a closed ideal in B.thus the quotient $B/(0:m)$ is a locally convex F-algebra.

We prove D is algebraicly isomorphic to $B/(0:m)$. Let $f \in D$, $f \neq 0$. Schur Lemma shows $fm \neq 0$. Thus there is $a_f \in A$ such that $fm = ma_f$ since $M = mA$. It is easy to show that $a_f \in B$ and $f \to a_f + (0:m)$ is an algebraic isomorphism from D onto $B/(0:m)$.

This isomorphism induces a topology on D which makes D a locally convex F-algebra. Since D is a division algebra (Schur Lemma), the desired conclusion follows, by Theorem 9.4 of [2].

Corollary 1. If A is complex and M is faithful, then A is strictly dense on the vector space M, i.e., given $x_1, \cdots, x_n$; $y_1, \cdots, y_n$ in M with $x_1, \cdots, x_n$ linearly independent, there is $a \in A$ such that $y_i = x_i a$ for $i = 1, 2, \cdots, n$.

Proof. By Lemma 5 and the Jacobson Density Theorem[1,p172].

If A is not complete, then Lemma 5 does not necessarily hold. [3] gave a commutative, locally convex, metrizable complex division algebra $C(t)$, which is not isomorphic to the complexes. $C(t)$ is itself a $C(t)$-faithful irreducible module and $\mathrm{End}_{C(t)}(C(t))$ contains $C(t)$ and hence is not isomorphic to the complexes. Another example of [4, pp141–146] shows Lemma 5 is not true for (not locally convex) F-algebras either.

Theorem 2 Let A be a locally finite (which means every finitely generated subalgebra is finite dimensional), primitive locally convex F-algebra and M an A-faithful irreducible module with the canonical topology separated. Then A is finite dimensional.

Proof. Suppose $\|\cdot\|_n$ is a sequence of seminorms on A giving the topology of A with $\|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant \|\cdot\|_3 \cdots$[2, p29]. Assuming A is not finite dimensional, then M, as a left vector space on the division algebra D, is infinite dimensional (since, by Lemma 5, D is a finite dimensional algebra on K, and M is an infinite dimensional K-vector space). Take any $0 \neq m \in$ M. We shall show by induction there is a sequence $m_0 = m$, m_1, m_2, $\cdots$ of D-linearly independent in M and a sequence x_1, x_2, $\cdots$ in A such that $m_r = mx_n^r (1 \leqslant r \leqslant n)$ and $\|x_{n-1} - x_n\|_n < 2^{-n}$ for all n. Since M is irreducible and has D-dimension greater than 1, there is $x_1 \in$ A such that $m_1 = mx_1$ is D-linearly independent of m. Suppose that m_0, m_1, $\cdots$, m_n and x_1, $\cdots$, x_n have been chosen. If $m_n x_n$ is not in the D-span of m_0, m_1, $\cdots$, m_n, put $x_{n+1} = x_n$ and $m_{n+1} = m_n x_n$. Otherwise, take any $m' \in$ M not in the span of m_0, $\cdots$, m_n. The Jacobson Density Theorem gives a $y \in$ A such that $m_i y = 0$ $(0 \leqslant i \leqslant n-1)$ and $m_n y = m'$. Put $x_{n+1} = x_n + \lambda y$ with $0 < |\lambda| < 2^{-n-1} / \|y\|_{n+1}$ so that $\|x_{n+1} - x_n\|_{n+1} < 2^{-n-1}$, and put $m_{n+1} = m_n x_{n+1}$. We have m_0, $\cdots$, m_{n+1} D-linearly independent and $mx_{n+1}^r = m_r (1 \leqslant r \leqslant n+1)$.

For any integer k, when $n > k$, we have

$$\|x_n - x_{n+p}\|_k \leqslant \sum_{i=1}^{p} \|x_{n+i-1} - x_{n+i}\|_k \leqslant \sum_{i=1}^{p} \|x_{n+i-1} - x_{n+i}\|_{n+i} \leqslant \sum_{i=1}^{p} 2^{-n-i} < 2^{-n}$$

for every integer p. So (x_n) is a Cauchy sequence in A and hence converges to some x in A. Since multiplication in F-algebras is jointly continuous[2, p23], we have $x_n^r \to x^r (r = 1, 2, \cdots)$ and, therefore, $mx^r = m_r (r = 1, 2, \cdots)$. Since the m_r's are K-linearly independent, the set $(x^1, x^2, \cdots)$ is linearly independent in A. This contradicts the local finiteness of A.

Ⅲ If A is a primitive algebra with minimal one-sided ideals, then there exists at most one topology on A which makes A a locally convex F-algebra. This is the main result of this section.

Rickart had proved this theorem for Banach algebras[5]. Our proof follows that of Rickart. From now on we suppose A is an algebra and P_1 and P_2 are two total paranorms on A such that both A$[P_1]$ and A$[P_2]$ are locally convex F-algebras. The Closed Graph Theorem shows that P_1 and P_2 are equivalent if and only if $x_n \xrightarrow{P_1} 0$ and $x_n \xrightarrow{P_2} s \in A$ implies $s=0$.

For $s \in A$, put $\Delta(s) = \inf\{P_1(x) + P_2(s-x);\ x \in A\}$. Let $\Delta = \{s \in A;\ \Delta(s) = 0\}$, then P_1 and P_2 are equivalent if and only if $\Delta = (0)$.

Lemma 6 Δ is a closed ideal in both $A[P_1]$ and $A[P_2]$.

Proof Let $s \in \Delta$, then there is a sequence $x_n \in A$ such that $P_1(x_n) \to 0$ and $P_2(s-x_n) \to 0$ as $n \to \infty$. For any $a \in A$, the continuity of multiplication shows $P_1(x_n a) \to 0$ and $P_2(sa - x_n a) \to 0$. So $sa \in \Delta$. Similarly, we have $as \in \Delta$. So Δ is an ideal of A.

Now suppose $s_n \to s$ in $A[P_2]$, $s_n \in \Delta$, then for every integer k there are $x_{nk} \in A$ such that

$$P_1(x_{nk}) + P_2(s_n - x_{nk}) < 1/k, \quad n = 1, 2, \cdots$$

So

$$P_1(x_{nn}) + P_2(s - x_{nn}) < P_1(x_{nn}) + P_2(s_n - x_{nn}) + P_2(s - s_n)$$
$$< P_2(s - s_n) + 1/n, \quad n = 1, 2, \cdots.$$

It follows that $P_1(x_{nn}) + P_2(s - x_{nn}) \to 0$ as $n \to \infty$. This implies $s \in \Delta$. So Δ is closed in $A[P_2]$. That Δ is closed in $A[P_1]$ can be proved same way.

The ideal Δ is called the separating ideal for P_1 and P_2.

Lemma 7 Let e be an idempotent ($i.e.$, $e^2 = e$) in A. Denote by Δ_1 the separating ideal for the restrictions on eAe of P_1 and P_2. Then $\Delta_1 = e\Delta e$.

Proof It is obvious that $\Delta_1 \subset \Delta$ and hence $\Delta_1 = e\Delta_1 e \subset e\Delta e$. On the other hand, if $s \in \Delta$ then there is a sequence $x_n \in A$ such that $P_1(x_n) \to 0$ and $P_2(s - x_n) \to 0$. Thus we have $P_1(ex_n e) \to 0$ and $P_2(ese - ex_n e) \to 0$ and hence $ese \in \Delta_1$. It follows that $\Delta_1 \supset e\Delta e$. So $\Delta_1 = e\Delta e$.

obviously, eAe is a closed subalgebra of $A[P_i]$ and hence also a locally convex F-algebra.

Theorem 3 Let S be the socle of A. Then both ΔS and $S\Delta$ are contained in the Jacobson radical R of A.

Proof Since S is the sum of the minimal right (left) ideals of A, it suffices to prove that $I\Delta \subset R$ for every minimal right ideal I. If $I^2 = (0)$ then $I \subset R$ and, therefore, $I\Delta \subset R$. Now suppose $I^2 \neq (0)$ then there exists an idempotent $e \in A$ such that $I = eA$ and eAe is a division algebra. As mentioned before $eAe[P_i]$ are locally convex F-algebras and so isomorphic to the reals or the complexes or the quaternions. Thus P_1 and P_2 are equivalent on eAe. So we have $e\Delta e = (0)$ by Lemma 7. This implies that $e\Delta = (0)$. Hence $I\Delta = eA\Delta \subset e\Delta = (0) \subset R$. The proof is completed.

Corollary 2 If A has no nonzero nilpotent one-sided ideals, then $\Delta S = S\Delta = (0)$.

Corollary 3. Let A be a primitive algebra with minimal onesided ideals. Then P_1 and P_2 are equivalent.

Proof. That $S\Delta=(0)$ implies $\Delta=(0)$ is obvious in this case.

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