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Initial Boundary Value Problem for One Class of System of Multidimensional Nonlinear Schrödinger-Boussinesq Type Eequations*

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§ 1. Introduction

In laser-plasma nonlinear interactions, the problems of solution for one class of system of nonlinear Schrödinger-Boussinesq type equations in one dimension have been studied in [1,2]. In [3] a class of multidimensional nonlinear Boussinesq type has been proposed. Under some conditions, we have proved the existence and uniquenss of the global solution for one class of system of nonlinear Schrödinger-Boussinesq type equations in one dimension in [4].

In this paper, we consider the following initial-boundary value problem for one class of the system multidimensional nonlinear Schrodinger-Boussinesq type equations

$$i\varepsilon_t+\Delta\vec{\varepsilon}-n\vec{\varepsilon}+\beta|\vec{\varepsilon}|^2\vec{\varepsilon}=0 \tag{1.1}$$

$$n_t=\Delta\varphi \tag{1.2}$$

$$\varphi_t=n+f(n)+\nu n_t-\lambda\Delta n+|\vec{\varepsilon}|^2 \tag{1.3}$$

$$\vec{\varepsilon}|_{t=0}=\vec{\varepsilon}_0(x),\ n|_{t=0}=n_0(x),\ \varphi|_{t=0}=\varphi_0(x),\ x\in\Omega \tag{1.4}$$

$$\vec{\varepsilon}|_{\partial\Omega}=n|_{\partial\Omega}=\varphi|_{\partial\Omega}=0 \tag{1.5}$$

where ν,λ are positive constants, β is a real constant; $i=\sqrt{-1}$; $\varepsilon(x,t)=(\varepsilon_1(x,t),\cdots,\varepsilon_N(x,t))$ is an unknown complex valued functional vector, $n(x,t)$ and $\varphi(x,t)$ are unknown real valued functions; $f(s)$ is a known complex valued real function, $s\in R^1$. $\vec{\varepsilon}_0(x)$ is a known complex valued functional vector, $n_0(x)$ and $\varphi_0(x)$ are known real valued functions. Let $\Omega\subset R^l$ be a smooth bounded domain and $\partial\Omega$ its boundary. $x=(x_1,\cdots,x_l)\in\Omega$, $\Delta\equiv\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_l^2}$. By means of the Galerkin method and the integral priori estimate method under some conditions we prove the existence and uniquence of the global solution for the problem (1.1)-(1.5).

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Here, we adopt the usual notation and convention. Let $H^m(\Omega)$ denote the Sobolev space with the norm $\|u\|_{H^m(\Omega)}=(\sum_{|\alpha|\le m}\|D^\alpha u\|^2_{L_2})^{\frac{1}{2}}$ or simply $\|u\|_m$; $H_0^m(\Omega)$ denote the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$; $\|u\|_{L_\infty}=\operatorname{ess}\sup_{x\in\Omega}|u(x)|$,

$\|\vec{u}\|_{H^m(\Omega)}=(\sum_{i=1}\|u_i\|^2_{H^m(\Omega)})^{\frac{1}{2}}$ and so on (see [5]).

§ 2. Existence of the Generalized Solutions of Problem (1.1)-(1.5)

We construct an approximate solution of the problem (1.1) - (1.5) by the Galerkin method, and choose a basis $\{w_j\}\subset H_0^1\cap H^2$, which are the eigenfunctions of the problem:

$$-\Delta w_j=\lambda_j w_j,\quad w_j|_{\partial\Omega}=0 \tag{2.1}$$

Obviously, if domain Ω is suitably smooth, then there must exist such a special basis. In fact, if $\Omega\in C^2$, then the basis $\{w_j\}\subset H^2(\Omega)\cap H_0'(\Omega)$, and it is dense in $H_0^1(\Omega)$.

Now suppose that the approximate solution be written as

$$\vec{\varepsilon}_m(x,t)=\sum_{j=1}^m a_{jm}(t)w_j(x),\quad n_m(x,t)=\sum_{j=1}^m \beta_{jm}(t)w_j(x),$$

$$\varphi_m(x,t)=\sum_{j=1}^m \gamma_{jm}(t)w_j(x) \tag{2.2}$$

where

$\vec{\varepsilon}_m(x,t)=(\varepsilon_{m1}(x,t),\varepsilon_{m2}(x,t),\cdots,\varepsilon_{mN}(x,t))$, $\vec{a}_{jm}(t)=(a_{jm1}(t),a_{jm2}(t),\cdots,a_{jmN}(t))$ are complex functional vectors. According to Galerkin's method, these coefficient vectors $a_{jm}(t)$ and coefficients $\beta_{jm}(t)$, $\gamma_{jm}(t)$ need to satisfy the following initial value problem of the system of the ordinary differential equations.

$$i(\varepsilon_{mlt},w_s)+(\Delta\varepsilon_{ml},w_s)-(n_m\varepsilon_{ml},w_s)+\beta\,(|\vec{\varepsilon}_m|^2\varepsilon_{ml},w_s)=0, \tag{2.3}$$

$$(n_{mt}-\Delta\varphi_m,w_s)=0; \tag{2.4}$$

$$(\varphi_{mt}-n_m-f(n_m)-\upsilon n_{mt}+\lambda\Delta n_m-|\vec{\varepsilon}_m|^2,w_s)=0, \tag{2.5}$$

$$\varepsilon_{ml}(x,0)=\varepsilon_{ml0}(x),\ n_m(x,0)=n_{m0}(x),\ \varphi_m(x,0)=\varphi_{m0}(x) \tag{2.6}$$

$$(l=1,2,\cdots,N;\ S=1,2,\cdots,m),$$

where $(u,v)=\int_\Omega u(x)\bar{v}(x)\,dx$, $\varepsilon_{ml}(x,t)=\sum_{j=1}^m a_{jml}(t)w_j(x)$, $|\vec{\varepsilon}_m|^2=\sum_{l=1}^N|\varepsilon_{ml}|^2$.

Suppose that

$$\varepsilon_{ml0}(x)\to\varepsilon_{0l}(x),\ n_{m0}(x)\to n_0(x),\ \varphi_{m0}(x)\to\varphi_0(x)\ \text{in } H_0'(\Omega),\ m\to\infty,\ l=1,2,\cdots,N. \tag{2.7}$$

Under the following conditions of Lemmas and priori estimates, we know that there exists a global solution in the interval for the initial value proplem (2.3) - (2.6) of the system of nonordinary differential equations and it can approximate the solution of the problem (1.1) - (1.5),

Lemma 1 If $\vec{\varepsilon}_0(x)\in L_2$, then for the solution $\vec{\varepsilon}_m(t)$ of the problem (2.3) - (2.6), we have

$$\|\vec{\varepsilon}_m(t)\|^2_{L_2\times L_\infty}\le E_0 \tag{2.8}$$

where the constant E_0 is independent of m.

Lemma 2 If $u(x)\in H^1_0(\Omega)$, $\Omega\subset R^2$, then there is the estimate

$$\|u(x)\|^4_{L_4}\le\frac{1}{2}\|\nabla u\|^2_{L_2}\|u\|^2_{L_2} \tag{2.9}$$

Lemma 3 If the following conditions are satisfied:

(i) $\beta<0$, $\nu>0$, $\lambda>0$, $\int_0^n f(z)dz\ge0$

(ii) $\|\vec{\varepsilon}_0(x)\|^2_{L_2}<1$, and $\vec{\varepsilon}_0(x), n_0(x), \varphi_0(x)\in H^1_0(\Omega)$, $x\in\Omega\subset R^2$, then there is the estimates

$$\|\nabla\vec{\varepsilon}_m(t)\|^2_{L_2\times L_\infty}+\|\nabla\varphi_m(t)\|^2_{L_2\times L_\infty}+\|\nabla n_m(t)\|^2_{L_2\times L_\infty}+\|n_m(t)\|^2_{L_2\times L_\infty}\le E_1, \tag{2.10}$$

where the constant E_1 is independent of m.

Proof Multiplying (2.3) by $a'_{sml}(t)$, and summing up for s from 1 to m, it follows

$$i(\varepsilon_{mlt},\varepsilon_{mlt})+(\Delta\varepsilon_{ml},\varepsilon_{mlt})-(n_m\varepsilon_{ml},\varepsilon_{mlt})+\beta(|\vec{\varepsilon}_m|^2\varepsilon_{ml},\varepsilon_{mlt})=0, \tag{2.11}$$

Taking the real part of (2.11), and summing up for l from 1 to N, it yields

$$\frac{d}{dt}\|\nabla\vec{\varepsilon}_m\|^2_{L_2}+\int_\Omega n_m|\vec{\varepsilon}_m|^2_t dx-\frac{\beta}{2}\frac{d}{dt}\int_\Omega|\vec{\varepsilon}_m|^4dx=0. \tag{2.12}$$

Since

$$\int_\Omega n_m|\vec{\varepsilon}_m|^2_t dx=\frac{d}{dt}\int_\Omega n_m|\vec{\varepsilon}_m|^2dx-\int_\Omega n_{mt}|\vec{\varepsilon}_m|^2dx,$$

and multiplying (2.5) by $\beta'_{sm}(t)$, and summing up for s from 1 to m, it follows

$$(\varphi_{mt}-n_m-f(n_m)-\nu n_{mt}+\lambda\Delta n_m-|\vec{\varepsilon}_m|^2,\ n_{mt})=0.$$

So

$$-(|\vec{\varepsilon}_m|^2,n_{mt})=(-\varphi_{mt},n_{mt})+(f(n_m),n_{mt})+\nu(n_{mt},n_{mt})-\lambda(\Delta n_{m1},n_{mt})+(n_m,n_{mt}). \tag{2.13}$$

Multiplying (2.4) by $\gamma'_{sm}(t)$, and summing up for s from 1 to m, it yields

$$(n_{mt}-\Delta\varphi_m,\ \varphi_{mt})=0.$$

Thus from

$$-(\varphi_{mt},n_{mt})=(\Delta\varphi_m,\ \varphi_{mt})=\frac{1}{2}\frac{d}{dt}\|\nabla\varphi_m\|^2_{L_2},$$

and

$$(n_m, n_{mt}) = \frac{1}{2}\frac{d}{dt}\| n_m \|_{L_2}^2, \qquad (f(n_m), n_{mt}) = \frac{d}{dt}\int_\Omega\int_0^{n_m} f(z)\,dz\,dx,$$

$$-\lambda(\Delta n_m, n_{mt}) = \frac{\lambda}{2}\frac{d}{dt}\| \nabla n_m \|_{L_2}^2 .$$

Substituting these relations into (2.12) and (2.13), there is

$$\frac{d}{dt}\| \nabla \vec{\varepsilon}_m \|_{L_2}^2 + \frac{d}{dt}\int_\Omega n_m |\vec{\varepsilon}_m|^2 dx + \frac{1}{2}\frac{d}{dt}\| \nabla \varphi_m \|_{L_2}^2 + \frac{1}{2}\frac{d}{dt}\| n_m \|_{L_2}^2 + \frac{d}{dt}\int_\Omega\int_0^{n_m} f(z)\,dz\,dx +$$

$$+ v\| n_{mt} \|_{L_2}^2 + \frac{\lambda}{2}\frac{d}{dt}\| \nabla n_m \|_{L_2}^2 - \frac{\beta}{2}\frac{d}{dt}\int_\Omega |\vec{\varepsilon}_m|^4 dx = 0 .$$

Integrating the above equality with respect to t, it follows

$$E(t) \equiv \| \nabla \vec{\varepsilon}_m \|_{L_2}^2 + \int_\Omega n_m |\vec{\varepsilon}_m|^2 dx + \frac{1}{2}\| \nabla \varphi_m \|_{L_2}^2 + \frac{1}{2}\| n_m \|_{L_2}^2 + \int_\Omega\int_0^{n_m} f(z)\,dz\,dx +$$

$$+ v\int_0^t \| n_{mt}(\tau) \|_{L_2}^2 d\tau + \frac{\lambda}{2}\| \nabla n_m \|_{L_2}^2 - \frac{\beta}{2}\int_\Omega |\vec{\varepsilon}_m|^4 dx = E(0) . \qquad (2.14)$$

Since $\int_0^{n_m} f(z)\,dz \geq 0$, using the inequality (2.9) in the following,

$$\left|\int_\Omega n_m |\vec{\varepsilon}_m|^2 dx\right| \leq \frac{1}{4}\| n_m \|_{L_2}^2 + \| \vec{\varepsilon}_m \|_{L_4}^4 \leq \frac{1}{4}\| n_m \|_{L_2}^2 + \frac{1}{2}\| \nabla \vec{\varepsilon}_m(t) \|_{L_2}^2 \| \vec{\varepsilon}_m(t) \|_{L_2}^2$$

$$= \frac{1}{4}\| n_m \|_{L_2}^2 + \frac{1}{2}\| \nabla \vec{\varepsilon}_m(t) \|_{L_2}^2 \| \vec{\varepsilon}_m(0) \|_{L_2}^2 \leq \frac{1}{4}\| n_m \|_{L_2}^2 + \| \nabla \vec{\varepsilon}_m(t) \|_{L_2}^2 \| \vec{\varepsilon}_0(x) \|_{L_2}^2 \quad (m \geq m_0)$$

hence from (2.14) it yields

$$(1 - \| \vec{\varepsilon}_0(x) \|_{L_2}^2)\| \nabla \vec{\varepsilon}_m \|_{L_2}^2 + \frac{1}{2}\| \nabla \varphi_m \|_{L_2}^2 + \frac{1}{4}\| n_m \|_{L_2}^2 + \int_\Omega\int_0^m f(z)\,dz\,dx$$

$$+ v\int_0^t \| n_{mt}(2) \|_{L_2}^2 dz + \frac{\lambda}{2}\| \nabla n_m(t) \|_{L_2}^2 - \frac{\beta}{2}\int_\Omega |\vec{\varepsilon}_m(t)|^4 dx \leq |E(0)| .$$

Thus the estimate (2.10) is obtained immediately.

Lemma 4 If the following conditions are satisfied:

(i) $\beta > 0$, $v > 0$, $\lambda > 0$, $\int_0^n f(z)\,dz \geq 0$

(ii) $(1 + \frac{\beta}{2})\| \vec{\varepsilon}_0(x) \|_{L_2}^2 < 1$, and $\vec{\varepsilon}_0(x) \in H_0^1(\Omega)$, $\varphi_0(x) \in H_0^1(\Omega)$.

then the estimate (2.10) is true too.

Lemma 5 Suppose that the conditions of Lemma 3 or Lemma 4 are satisfied, and $|f(z)| \leq A|z|^q + B$ holds, where A, B are positive constants, $1 \leq q < \infty$, then we have

$$\| \vec{\varepsilon}_{mt} \|_{H^{-1}(\Omega) \times L^\infty(0,T)} + \| \varphi_{mt} \|_{H^{-1}(\Omega) \times L^\infty(0,T)} + \| n_{mt} \|_{H^{-1}(\Omega) \times L^\infty(0,T)} \leq E_2 \qquad (2.15)$$

where the constant E_2 is independent of m.

Lemma 6 If the conditions of Lemma 3 or Lemma 4 are satisfied, then we have

$$\|\vec{\varepsilon}_m(\cdot,t+\Delta t)-\vec{\varepsilon}_m(\cdot,t)\|_{L_2}+\|n_m(\cdot,t+\Delta t)-n_m(\cdot,t)\|_{L_2}+\|\varphi_m(\cdot,t+\Delta t)$$
$$-\varphi_m(\cdot,t)\|_{L_2}\leq C\Delta t^{\frac{1}{2}} \tag{2.16}$$

where the constant C is independent of m.

Difinition I The functions $\{\vec{\varepsilon}(x,t),n(x,t),\varphi(x,t)\}$ is called the generalized solution on the region $[0,\infty)\times\Omega$ for the initial-boundary value problem (1.1) – (1.5), if the following conditions are satisfied by $\{\vec{\varepsilon}(x,t),$ $n(x,t),\varphi(x,t)\}$:

(i) $\vec{\varepsilon}(x,t),\ n(x,t),\varphi(x,t)\in L^\infty(0,T;H_0'(\Omega))\cap w_\infty^{(1)}(0,T;H^{-1}(\Omega))\cap$

$$\cap C^{(0,\frac{1}{2})}(0,T;L_2(\Omega))$$

(ii) $$\int_0^T[i(\vec{\varepsilon}(t),\Sigma_t(t))+((\vec{\varepsilon}(t),\Sigma(t)))+(n\vec{\varepsilon}(t),\Sigma(t))-\beta(|\vec{\varepsilon}|^2\vec{\varepsilon},\Sigma(t))]dt$$
$$+i(\vec{\varepsilon}(0),\Sigma(0))=0 \tag{2.17}$$

where $\Sigma(t)=\Sigma(x,t)$ is any complex valued function,

$$\Sigma(t)\in C^1(0,T;L_2(\Omega))\cap C^0(0,T;H_0^1(\Omega)\cap H^2(\Omega)),\ \text{and}\ \Sigma(T)=0.$$

and $$((u,v))=\int_\Omega\nabla u\cdot\nabla\bar{v}\,dx.$$

(iii) $$\int_0^T[(n(t),N_t(t))+((\varphi(t),N(t)))]dt+(n(0),N(0))=0 \tag{2.18}$$

Here $N(t)=N(x,t)$ is any real valued function,

$$N(t)\in C^1(0,T;L_2(\Omega))\cap C^0(0,T;H_0^1(\Omega)),\ \text{and}\ N(T)=0$$

(iv) $$\int_0^T[(\varphi(t),\Phi_t(t))+(n(t)+f(n)+|\vec{\varepsilon}|^2,\Phi(t))-v(n(t),\Phi_t(t))$$
$$+\lambda((n(t),\Phi(t)))]dt+(\varphi(0),\phi(0))-v(n(0),\Phi(0))=0 \tag{2.19}$$

Here $\Phi(x)$ is any real valued function

$$\Phi(x)\in C^1(0,T;L_2(\Omega))\cap C^0(0,T;L_2(\Omega)),\ \text{and}\ \Phi(T)=0$$

Theorem I Suppose that the conditions of Lemma 5 are satisfied, then there exists the generalized solution for the initial-boundary value problem (1.1) – (1.5).

§3 Existence of the strong solution for the problem (1.1)–(1.5)

Lemma 7 If the following conditions are satisfied

(i) $v>0,\ \lambda>0,\int_0^n f(z)dz\geq 0$

(ii) $\|\vec{\varepsilon}_0(x)\|_{L_2}^2<1$, and $\vec{\varepsilon}_0(x),\ n_0(x),\varphi_0(x)\in H_0^1(\Omega),\ x\in\Omega\subseteq R^2$. Then for the solution of the problem (1.1) – (1.5) ($\beta=0$), there is the estimate

$$\|\nabla\varepsilon\|^2_{L_2\times L_\infty}+\|\nabla\varphi\|^2_{L_2\times L_\infty}+\|\nabla n\|^2_{L_2\times L_\infty}+\|n\|^2_{L_2\times L_\infty}\le E_3 \qquad (3.1)$$

where the constant E_3 only depends on the initial value functions $\vec{\varepsilon_0}(x)$, $n_0(x)$, $\varphi_0(x)$ and their first order derivatives.

Lemma 8 For the solution $\vec{\varepsilon}(x_1, x_2, t)$ of problem (1.1) – (1.5) ($\beta=0$), we have estimate

$$\|\nabla\vec{\varepsilon}\|_{L_p}\le C_T\ (p>2),\quad \|\vec{\varepsilon}\|_{L_\infty}\le C_T,\quad (p>2) \qquad (3.2)$$

where C_T is a definite constant.

Proof The solution $\vec{\varepsilon}(x,t)$ of (1.1) can be expressed as

$$\vec{\varepsilon}(t)=s(t)\vec{\varepsilon_0}+\int_0^t s(t-\xi)[n(\xi)\vec{\varepsilon}(\xi)]d\xi,$$

where $s(t)\varphi=e^{it\Delta}\varphi$ is a semigroup generated by the operator Δ. Differentiating the above equality with respect to the sapce variables, we have

$$D\vec{\varepsilon}(t)=s(t)D\vec{\varepsilon_0}+\int_0^t s(t-\xi)D[n(\xi)\vec{\varepsilon}(\xi)]d\xi,$$

$$\|D\vec{\varepsilon}(t)\|_{L_p}\le\|s(t)D\vec{\varepsilon_0}\|_{L_p}+c\int_0^t (t-\xi)^{-\frac{(2-q)}{q}}\|nD\vec{\varepsilon}+Dn\vec{\varepsilon}\|_{L_q}d\xi$$

$$\le C_1\|\vec{\varepsilon_0}\|_{H^2}+C\int_0^t(t-\xi)^{-\frac{(2-q)}{q}}[\|n\|_{L_r}\|D\vec{\varepsilon}\|_{L_2}+\|\vec{\varepsilon}\|_{L_r}\|Dn\|_{L_2}]d\xi$$

$$\le C_1\|\vec{\varepsilon_0}\|_{H^2}+C_2\int_0^t(t-\xi)^{\frac{(2-q)}{q}}d\xi\le C_3(t)\le C_T.$$

where $\frac{1}{p}+\frac{1}{q}=1$, and $\frac{1}{r}=\frac{1}{2}-\frac{1}{p}=\frac{p-2}{2p}$, as $p>2$, we can ascertain that $\|n\|_{L_r}$ and $\|\vec{\varepsilon}\|_{L_r}$ are bounded. For example, if we take $p=3$, then $r=5$. By Lemma 7, $\|\nabla\vec{\varepsilon}\|_{L_2}$ and $\|\nabla n\|_{L_2}$ are bounded. From Sobolev's inequality $\|\vec{\varepsilon}\|_{L_6}$ and $\|n\|_{L_6}$ are bounded. Hence for $p>2$, Consequenty, and by Sobolev's inequality we have (3.2)

Lemma 9 Suppose that the conditions of Lemma 3 are satisfied, and assume that

(i) $f(n)\in C^2$, $|f'(n)|\le A|n|^{q-1}$, $A>0$, $q\ge1$.

(ii) $\vec{\varepsilon_0}(x)\in H_0^1(\Omega)\cap H^2(\Omega)$, $n_0(x)\in H_0^1(\Omega)\cap H^2(\Omega)$, $\varphi_0(x)\in H_0^1(\Omega)\cap H^2(\Omega)$.

Then for the solution of problem (1.1) – (1.5), there is the estimations

$$\|\vec{\varepsilon}_t\|^2_{L_2\times L_\infty}+\|n_t\|^2_{L_2\times L_\infty}+\|\Delta n\|^2_{L_2\times L_\infty}+\int_0^T\|\Delta n_t(\tau)\|^2_{L_2}d\tau\le E_4, \qquad (3.3)$$

where the constant E_4 depends on the norms of $\|\vec{\varepsilon_0}\|_{H^2}$, $\|n_0(x)\|_{H^2}$ and $\|\varphi_0(x)\|_{H^2}$.

Proof From (1.3), it follows

$$\Delta\varphi_t - \Delta n - \Delta f(n) - \nu\Delta n_t + \lambda\Delta^2 n - \Delta|\vec{\varepsilon}|^2 = 0. \tag{3.4}$$

Differentiating (1.2) with respect to t, it yields

$$n_{tt} - \Delta\varphi_t = 0 \tag{3.5}$$

Put (3.5) into (3.4), we obtain

$$n_{tt} - \Delta n - \Delta f(n) - \nu\Delta n_t + \lambda\Delta^2 n - \Delta|\vec{\varepsilon}|^2 = 0. \tag{3.6}$$

Taking the inner product for (3.6) with n_t, it follows

$$(n_{tt} - \Delta n - \Delta f(n) - \nu\Delta n_t + \lambda\Delta^2 n - \Delta|\vec{\varepsilon}|^2, n_t) = 0. \tag{3.7}$$

Since

$$(n_{tt}, n_t) = \frac{1}{2}\frac{d}{dt}\|n_t\|^2_{L_2}, \quad (-\Delta n, n_t) = \frac{1}{2}\frac{d}{dt}\|\nabla n\|^2_{L_2},$$

$$-(\nu\Delta n, n_t) = \nu\|\nabla n_t\|^2_{L_2}, \quad \lambda(\Delta^2 n, n_t) = \frac{\lambda}{2}\frac{d}{dt}\|\Delta n\|^2_{L_2},$$

$$|(-\Delta f(n), n_t)| \le \|\nabla f(n)\|_{L_2}\|\nabla n_t\|_{L_2} \le \frac{3}{4\nu}\|\nabla f(n)\|^2_{L_2} + \frac{\nu}{3}\|\nabla n_t\|^2_{L_2},$$

$$\|\nabla f(n)\|^2_{L_2} \le \|f'(n)\|^2_{L_{2(2+\delta)/\delta}} \|\nabla n\|^2_{L_{2+\delta}} \ (\delta > 0) \le A^2\|n\|^{2(q-1)}_{L_{2(q-1)(2+\delta)/\delta}}\|\nabla n\|^2_{L_{2+\delta}}$$

$$\le C\|\nabla n\|^{2(q-1)}_{L_2}\|\nabla n\|^2_{L_{2+\delta}} \le C_1\|\nabla n\|^2_{L_{2+\delta}} \le C_2\|D^2 n\|^{\frac{2(\delta+1)}{2+\delta}}_{L_2}\|n\|^{\frac{2}{2+\delta}}_{L_2},$$

thus

$$\frac{3}{4}\|\nabla f(n)\|^2_{L_2} \le C_4\|\Delta n\|^{\frac{2(\delta+1)}{2+\delta}}_{L_2} \le C_5\|\Delta n\|^2_{L_2} + C_6. \tag{3.8}$$

Now we are going to estimate the term

$$|(-\Delta|\vec{\varepsilon}|^2, n_t)| = |(\nabla|\vec{\varepsilon}|^2, \nabla n_t)| \le \frac{\nu}{3}\|\nabla n_t\|^2_{L_2} + \frac{3}{4\nu}\|\nabla|\vec{\varepsilon}|^2\|^2_{L_2}$$

$$\le \frac{\nu}{3}\|\nabla n_t\|^2_{L_2} + \frac{3}{\nu}\|\nabla\vec{\varepsilon}\|^2_{L_2}\|\vec{\varepsilon}\|^2_{L_\infty} \le \frac{\nu}{3}\|\nabla n_t\|^2_{L_2} + C_7. \tag{3.9}$$

Differentiating (1.1) with respect to t and taking the inner product for the resulting relation with $\vec{\varepsilon}_t$, we have

$$(i\vec{\varepsilon}_{tt}, \vec{\varepsilon}_t) + (\Delta\vec{\varepsilon}_t, \vec{\varepsilon}_t) - (n_t\vec{\varepsilon} + n\vec{\varepsilon}_t, \vec{\varepsilon}_t) = 0. \tag{3.10}$$

Taking the imaginary part in (3.10), it follows

$$\frac{1}{2}\frac{d}{dt}\|\vec{\varepsilon}_t\|^2_{L_2} \le |(n_t\vec{\varepsilon}, \vec{\varepsilon}_t)| \le \|\vec{\varepsilon}\|_{L_\infty}\frac{1}{2}(\|n_t\|^2_{L_2} + \|\vec{\varepsilon}_t\|^2_{L_2}) \le C_8(\|n_t\|^2_{L_2} + \|\varepsilon_t\|^2_{L_2}). \tag{3.11}$$

By (3.7)(3.8)(3.9)(3.11) we obtain

$$\frac{d}{dt}[\|n_t\|^2_{L_2} + \|\nabla n\|^2_{L_2} + \|\Delta n\|^2_{L_2} + \|\vec{\varepsilon}_t\|^2_{L_2}] + \|\nabla n_t\|^2_{L_2} \le C_9[\|n_t\|^2_{L_2} + \|\Delta n\|^2_{L_2} + \|\vec{\varepsilon}_t\|^2_{L_2}] + C_{10}.$$

By using Gronwall's inequality, (3.3) can be derived from the above inequality.

Lemma 10 Suppose that the conditions of Lemma 9 are satisfied, and assume that

(i) $f(n)\in C^3$,

(i i) $\vec{\varepsilon}_0(x)\in H^4\cap H_0^1, n_0(x)\in H^4\cap H_0^1$, $\varphi_0(x)\in H^4\cap H_0^1$.

Then we have

$$\|\Delta\vec{\varepsilon}_t\|^2_{L_2\times L_\infty}+\|n_{tt}\|^2_{L_2\times L_\infty}+\|\vec{\varepsilon}_{tt}\|^2_{L_2\times L_\infty}+\|\nabla n_{tt}\|^2_{L_2\times L_\infty}+\|\Delta^2 n\|^2_{L_2\times L_\infty}+\|\Delta^2\vec{\varepsilon}\|^2_{L_2\times L_\infty}$$
$$+\|\Delta n_t\|^2_{L_2\times L_\infty}\leq E_5 , \tag{3.12}$$

where E_5 is a definite constant.

Lemma 11. If the following coddi tions are satisfied

(i) $\beta=0$, $\nu>0$, $\lambda>0$, $\int_0^n f(z)dz>0$

(i i) $\|\vec{\varepsilon}_0(x)\|^2_{L_2}<1$, and $\vec{\varepsilon}_0(x), n_0(x)$, $\varphi_0(x)\in H_0^1(\Omega)$, $x\in\Omega\subset R^2$,

then for the solution of the problem (2.3)–(2.6), there is the estimates

$$\|\vec{\varepsilon}_m(t)\|^2_{H^1\times L_\infty}+\|\varphi_m(t)\|^2_{H^1\times L_\infty}+\|n_m(t)\|^2_{H^1\times L_\infty}\leq E_6 \tag{3.13}$$

where the constant E_6 is independent of m.

Lmma 12 Suppose the conditions of Lemma 9 are satisfied and assum assume that

$$\vec{\varepsilon}_{m0}(x)\xrightarrow{H^2}\vec{\varepsilon}_0(x),\quad n_{m0}(x)\xrightarrow{H^2}n_0(x),\quad \varphi_{m0}(x)\xrightarrow{H^2}\varphi_0(x).$$

Then for the solution of problem (2.3)–(2.6) there is a time interval $[0,t_0]$, such that the estimates

$$\|\vec{\varepsilon}_{mt}(t)\|^2_{L_2\times L_\infty}+\|n_{mt}\|^2_{L_2\times L_\infty}+\|\Delta n_m\|^2_{L_2\times L_\infty}+\int_0^{t_0}\|\Delta n_{mt}(\tau)\|^2_{L_2}d\tau\leq E_7 \tag{3.14}$$

hold, where the constant E_7 is independent of m, which depends on the norms of $\|\vec{\varepsilon}_0\|_{H^2}$, $\|n_0(x)\|_{H^2}$ and $\|\varphi_0(x)\|_{H^2}$.

Lemma 13 Suppose that the conditions of Lemma 12 are satisfied and assume that

(i) $f(n)\in C^3$

(i i) $\vec{\varepsilon}_0(x)\in H^4\cap H_0^1$, $n_0(x)\in H^4\cap H_0^1$, $\varphi_0(x)\in H^4\cap H_0^1$, and

$$\vec{\varepsilon}_{m/0}(x)\xrightarrow{H^4}\vec{\varepsilon}_0(x),\quad n_{m0}(x)\xrightarrow{H^4}n_0(x)\quad \varphi_{m0}(x)\xrightarrow{H^4}\varphi_0(x).$$

Then there is a time interval $[0,t_0]$, such that

$$\|\Delta\vec{\varepsilon}_{mt}\|^2_{L_2\times L_\infty}+\|n_{mtt}\|^2_{L_2\times L_\infty}+\|\vec{\varepsilon}_{mtt}\|^2_{L_2\times L_\infty}+\|\nabla n_m(t)\|^2_{L_2\times L_\infty}$$
$$+\|\Delta^2 n_m\|^2_{L_2\times L_\infty}+\|\Delta^2\vec{\varepsilon}_m\|^2_{L_2\times L_\infty}+\|\Delta n_{mt}\|^2_{L_2\times L_\infty}\leq E_8 , \tag{3.15}$$

where the constant E_8 is independent of m and t_0, only depends on the norms $\|\vec{\varepsilon}_0(x)\|_{H^4}$, $\|n_0(x)\|_{H^4}$ and $\|\varphi_0(x)\|_{H^4}$.

Theorem 2 (Local existence theorem) Suppose that the following conditions are satisfied

(i) $f(n)\in C^3$, $\int_0^n f(z)dz\geq 0$, $|f'(n)|\leq A|n|^{q-1}$, $q\geq 0$, $A>0$,

(ii) $\nu>0$, $\lambda>0$, $\|\vec{\varepsilon}_0(x)\|_{L_2}^2<1$, $x=(x_1,x_2)\in\Omega\subset R^2$,

(iii) $\vec{\varepsilon}_0(x)\in H^4\cap H_0^1$, $n_0(x)\in H^4\cap H_0^1$, $\varphi_0(x)\in H^4\cap H_0^1$.

Then there exists a local solution on $[0,t_0]$ for the problem (1.1)-(1.5) ($\beta=0$)

$\vec{\varepsilon}(x,t)\in L^\infty(0,t_0;H^4\cap H_0^1)$, $\vec{\varepsilon}_t(x,t)\in L^\infty(0,t_0;H^2\cap H_0^1)$, $\vec{\varepsilon}_{tt}(x,t)\in L^\infty(0,t_0;L_2)$;

$n(x,t)\in L^\infty(0,t_0;H^4\cap H_0^1)$, $n_t(x,t)\in L^\infty(0,t_0;H^2\cap H_0^1)$, $n_{tt}(x,t)\in L^\infty(0,t_0;H_0^1)$,

$\varphi(x,t)\in L^\infty(0,t_0;H^4\cap H_0^1)$, $\varphi_t(x,t)\in L^\infty(0,t_0;H^2\cap H_0^1)$, $\varphi_{tt}(x,t)\in L^\infty(0,t_0;H_0^1)$;

where the constant t_0 depends on the norms $\|\vec{\varepsilon}_0(x)\|_{H^4}$, $\|n_0(x)\|_{H^4}$ and $\|\varphi_0(x)\|_{H^4}$.

Difinition 2 The functions $\{\vec{\varepsilon}(x,t),n(n,t),\varphi(x,t)\}$ is called the globlal strong solution on the region $[0,\infty)\times\Omega$ for the initial-boundary value problem (1.1)-(1.5) ($\beta=0$), if the following conditions are satisfied by $\{\vec{\varepsilon}(x,t),n(x,t),\varphi(x,t)\}$:

(i) $\vec{\varepsilon}(x,t)\in L^\infty(0,T;H^4\cap H_0^1)$, $\vec{\varepsilon}_t(x,t)\in L^\infty(0,T;H^2\cap H_0^1)$, $\vec{\varepsilon}_{tt}(x,t)\in L^\infty(0,T;L_2)$;
$n(x,t)\in L^\infty(0,T;H^4\cap H_0^1)$, $n_t(x,t)\in L^\infty(0,T;H^2\cap H_0^1)$, $n_{tt}(x,t)\in L^\infty(0,T;H_0^1)$;
$\varphi(x,t)\in L^\infty(0,T;H^4\cap H_0^1)$, $\varphi_t(x,t)\in L^\infty(0,T;H^2\cap H_0^1)$, $\varphi_{tt}(x,t)\in L^\infty(0,T;H_0^1)$.

(ii) For any test function $v(x)\in H_0^1\cap H^2$, the following integral relations hold

$(i\vec{\varepsilon}_t+\Delta\vec{\varepsilon}-n\vec{\varepsilon},v)=0$, $(n_t-\Delta\varphi,v)=0$, $(\varphi_t-n-f(n)-\nu n_t+\lambda\Delta n-|\vec{\varepsilon}|^2,v)=0$.

(iii) $\vec{\varepsilon}|_{t=0}=\vec{\varepsilon}_0(x)$, $n|_{t=0}=n_0(x)$, $\varphi|_{t=0}=\varphi_0(x)$.

Theorem 3 Suppose that the conditions of Theorem 2 are satisfied, then there exists a global strong solution for the problem (1.1)-(1.5) ($\beta=0$)

Theorem 4 (Uniqueness theorem of smooth solution) Suppose that (i) $\vec{\varepsilon}_0(x)\in H^2\cap H_0^1$, $n_0(x)\in H^2\cap H_0^1$, and $\varphi(x)\in H^2\cap H_0^1$; (ii) $f(n)\in C^4$. Then the smooth solution $\vec{\varepsilon}(x,t)\in L^\infty(0,T;C^2)$, $n(x,t)\in L^\infty(0,T;C^2)$, $\varphi(x,t)\in L^\infty(0,T;C^2)$ for the problem (1.1)-(1.5) is unique.

Proof Suppose that there are two smooth solution: $\{\vec{\varepsilon}_1,n_1,\varphi_1\}$ and $\{\vec{\varepsilon}_2,n_2,\varphi_2\}$ for the problem, (1.1)-(1.5). Let $\vec{\varepsilon}=\vec{\varepsilon}_1-\vec{\varepsilon}_2$, $n=n_1-n_2$, $\varphi=\varphi_1-\varphi_2$. Then from (1.1) (1.2) (1.3), it follows

$$(i\vec{\varepsilon}_t+\Delta\vec{\varepsilon}-(n\vec{\varepsilon}_1+n_2\vec{\varepsilon}),w_j)=0 \tag{3.15}$$

$$(n_t-\Delta\varphi,\ w_j)=0 \tag{3.16}$$

$$(\varphi_t-n-(f(n_1)-f(n_2))-\nu n_t+\lambda\Delta n-(|\vec{\varepsilon}_1|^2-|\vec{\varepsilon}_2|^2),w_j)=0,\ \forall\, w_j\in H_0^1\cap H^2 \tag{3.17}$$

Differentiating (3.16) with respect to t, and taking $-\Delta w_j=\lambda_j w_j$ with (3.17), it follows

$$(n_{tt}-\Delta n-\Delta(f(n_1)-f(n_2))-\nu\Delta n_t+\lambda\Delta^2 n-\Delta(|\vec{\varepsilon_1}|^2-|\vec{\varepsilon_2}|^2),w_j)=0,\quad (3.18)$$

where

$$\Delta(f(n_1)-f(n_2))=(f''(n_1)-f''(n_2))|\nabla n_1|^2+f''(n_2)(|\nabla n_1|^2-|\nabla n_2|^2)+(f'(n_1)-f'(n_2))\Delta n_1+f'(n_2)(\Delta n_1-\Delta n_2),$$

$$\Delta(|\vec{\varepsilon_1}|^2-|\vec{\varepsilon_2}|^2)=\sum_{j=1}^{N}(\Delta\varepsilon_{1j}-\Delta\varepsilon_{2j})\bar{\varepsilon}_{1j}+\sum_{j=1}^{N}\Delta\varepsilon_{2j}(\bar{\varepsilon}_{1j}-\bar{\varepsilon}_{2j})$$
$$+2\sum_{j=1}^{N}[|\nabla\varepsilon_{1j}|^2-|\nabla\varepsilon_{2j}|^2]+\sum_{j=1}^{N}\Delta\bar{\varepsilon}_{1j}(\varepsilon_{1j}-\varepsilon_{2j})+\sum_{j=1}^{N}(\Delta\bar{\varepsilon}_{1j}-\Delta\bar{\varepsilon}_{2j})\varepsilon_{2j}.$$

Taking the inner product for (3.18) with n_t, we have

$$\frac{d}{dt}\frac{1}{2}[\|n_t\|_{L_2}^2+\|\nabla n\|_{L_2}^2+\lambda\|\Delta n\|_{L_2}^2]+\nu\|\nabla n_t\|_{L_2}^2$$
$$\leq\|f'''(\theta n)\|_{L_\infty}\|\nabla n_1\|_{L_\infty}^2\cdot\frac{1}{2}(\|n\|_{L_2}^2+\|n_t\|_{L_2}^2)+\|f''(n_2)\|_{L_\infty}(\|\nabla n_1\|_{L_\infty}+\|\nabla n_2\|_{L_\infty})$$
$$\times\frac{1}{2}(\|\nabla n\|_{L_2}^2+\|n_t\|_{L_2}^2)+\|f''\|_{L_\infty}\|\Delta n_1\|_{L_\infty}\cdot\frac{1}{2}(\|n\|_{L_2}^2+\|n_t\|_{L_2}^2)+\frac{1}{2}\|f'\|_{L_\infty}$$
$$\times(\|\Delta n\|_{L_2}^2+\|n_t\|_{L_2}^2)+\sum_{j=1}^{N}\|\varepsilon_{1j}\|_{L_\infty}\frac{1}{2}(\|\Delta\varepsilon_j\|_{L_2}^2+\|n_t\|_{L_2}^2)+\sum_{j=1}^{N}\|\Delta\varepsilon_j\|_{L_\infty}\times\frac{1}{2}$$
$$\times(\|\varepsilon_j\|_{L_2}^2+\|n_t\|_{L_2}^2)+2\sum^{N}(\|\nabla\varepsilon_{1j}\|_{L_\infty}+\|\nabla\varepsilon_{2j}\|_{L_\infty})\frac{1}{2}(\|\nabla\varepsilon_j\|_{L_2}^2+\|n_t\|_{L_2}^2)$$
$$+\sum_{j=1}^{N}\|\Delta\varepsilon_{1j}\|_{L_\infty}\times\frac{1}{2}(\|\varepsilon_j\|_{L_2}^2+\|n_t\|_{L_2}^2)+\sum_{j=1}^{N}\|\varepsilon_{2j}\|_{L_\infty}\frac{1}{2}(\|\Delta\varepsilon_j\|_{L_2}^2+\|n_t\|_{L_2}^2$$
$$\leq C[\|n\|_{L_2}^2+\|\nabla n\|_{L_2}^2+\|\Delta n\|_{L_2}^2+\|\Delta\vec{\varepsilon}\|_{L_2}^2+\|\vec{\varepsilon}\|_{L_2}^2+\|\nabla\vec{\varepsilon}\|_{L_2}^2+\|n_t\|_{L_2}^2].\quad (3.19)$$

Taking the inner product for (3.15) with $\Delta^2\vec{\varepsilon}$, it follows

$$(i\Delta\vec{\varepsilon_t}+\Delta^2\vec{\varepsilon}-\Delta(n\vec{\varepsilon_1}+n_2\vec{\varepsilon}),\Delta\vec{\varepsilon})=0,\quad (3.20)$$

where

$$\Delta(n\vec{\varepsilon_1}+n_2\vec{\varepsilon})=\Delta n\vec{\varepsilon_1}+2\nabla n\cdot\nabla\vec{\varepsilon_1}+n\nabla\vec{\varepsilon_1}+\Delta n_2\vec{\varepsilon}+2\nabla n_2\cdot\nabla\vec{\varepsilon}+n_2\Delta\vec{\varepsilon}.$$

Taking the imaginary part in (3.20), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta\vec{\varepsilon}\|_{L_2}^2\leq|(\Delta n\vec{\varepsilon_1},\Delta\vec{\varepsilon})|+2|(\nabla n\cdot\nabla\vec{\varepsilon_1},\Delta\vec{\varepsilon})|$$
$$+|(n\Delta\vec{\varepsilon_1},\Delta\vec{\varepsilon})|+|(\Delta n_2\vec{\varepsilon},\Delta\vec{\varepsilon})|+2|(\nabla n_2\cdot\nabla\vec{\varepsilon},\Delta\vec{\varepsilon})|$$
$$\leq\|\vec{\varepsilon_1}\|_{L_\infty}\cdot\frac{1}{2}(\|\Delta n\|_{L_2}^2+\|\Delta\vec{\varepsilon}\|_{L_2}^2)+\|\nabla\vec{\varepsilon_1}\|_{L_\infty}(\|\nabla n\|_{L_2}^2+\|\Delta\vec{\varepsilon}\|_{L_2}^2)$$
$$+\|\Delta\vec{\varepsilon_1}\|_{L_\infty}\cdot\frac{1}{2}(\|n\|_{L_2}^2+\|\Delta\vec{\varepsilon}\|_{L_2}^2)+\|\Delta n_2\|_{L_\infty}\cdot\frac{1}{2}(\|\vec{\varepsilon}\|_{L_2}^2+\|\Delta\vec{\varepsilon}\|_{L_2}^2)+2\|\nabla n_2\|_{L_\infty}\|\Delta\vec{\varepsilon}\|_{L_2}^2$$
$$\leq C[\|\Delta n\|_{L_2}^2+\|\Delta\vec{\varepsilon}\|_{L_2}^2+\|\nabla n\|_{L_2}^2+\|\vec{\varepsilon}\|_{L_2}^2].\quad (3.21)$$

Taking the inner product for (3.15) with $\vec{\varepsilon}$, we get

$$\frac{1}{2}\frac{d}{dt}\|\vec{\varepsilon}\|_{L_2}^2\leq|(n\vec{\varepsilon_1},\vec{\varepsilon})|\leq\|\vec{\varepsilon_1}\|_{L_\infty}\cdot\frac{1}{2}(\|n\|_{L_2}^2+\|\vec{\varepsilon}\|_{L_2}^2)\leq C(\|n\|_{L_2}^2+\|\vec{\varepsilon}\|_{L_2}^2) \tag{3.22}$$

and

$$\frac{d}{dt}\|n\|_{L_2}^2\leq\|n\|_{L_2}^2+\|n_t\|_{L_2}^2. \tag{3.23}$$

By using the following inequality

$$\|\nabla\vec{\varepsilon}\|_{L_2}^2\leq a\|\Delta\vec{\varepsilon}\|_{L_2}^2+a^{-\frac{1}{2}}C\|\vec{\varepsilon}\|_{L_2}^2 \quad (a>0,\ C>0)$$

and from (3.19)(3.21)(3.22) and (3.23), we obtain

$$\frac{d}{dt}[\|n_t\|_{L_2}^2+\|\nabla n\|_{L_2}^2+\|n\|_{L_2}^2+\|\vec{\varepsilon}\|_{L_2}^2+\|\Delta\vec{\varepsilon}\|_{L_2}^2]$$
$$\leq C[\|n_t\|_{L_2}^2+\|n\|_{L_2}^2+\|\nabla n\|_{L_2}^2+\|\Delta n\|_{L_2}^2+\|\vec{\varepsilon}\|_{L_2}^2+\|\Delta\vec{\varepsilon}\|_{L_2}^2]. \tag{3.24}$$

By the zero initial conditions $n|_{t=0}\equiv n_t|_{t=0}\equiv\nabla n|_{t=0}\equiv\Delta n|_{t=0}\equiv\vec{\varepsilon}|_{t=0}\equiv\Delta\vec{\varepsilon}|_{t=0}\equiv 0$, and (3.24) implies $n(x,t)\equiv\vec{\varepsilon}(x,t)\equiv 0$, (3.17) implies $\varphi(x,t)\equiv 0$. Hence the proof of theorem is completed.

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