J. Math, Res. & Exposition, 1988 No. 1

On the Naive Mathematical Models of Medium Mathematical System MM*

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Introduction

The aim of this paper is to explain the practical background and the ideological principles of succeed papers related to medium logic calculus ML and medium axiomatic set theory MS on the basis of reference [1]. The succeed papers, which include 15 papers, cover propositional calculus systems of medium logic $MP_{(I)-(III)}$ and their extention $MP^*_{(I)-(II)}$, predicate calculus systems of medium logic $MF_{(I)-(II)}$ and their extention $MF^*_{(I)}$, identity and diversity calculus system of medium logic $ME^*_{(I)}$, and medium axiomatic set theory $MS_{(I)-(IV)}$ etc., and of medium mathematical system MM shown by these papers can be figured:

$$\begin{tikzcd}[row sep=small] & & MP,\ MP^* \\ & ML \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & MF,\ MF^* \\ MM \arrow[ur, dash] \arrow[dr, dash] & & ME^* \\ & MS & \end{tikzcd}$$

the structure where MM is a short symbol of medium mathematics, so are ML, MP, MF, ME, MP*, MF*, and MS.

Through the discussion on a series of basic concepts and principles of medium objects, fuzzy predicates, non-medium principle, comprehension principle and pan comprehension principle etc., the paper explains the intuition thought and the objective prototype of MM and clarifies the aim and significance of forming MM. Hence, in a sense, the paper is an introduction to MM(ML&MS), and can also be regarded as a summary of above 15 papers.

§ I Fuzzy Predicates and Medium Objects in the Sense of Inverse Opposites

Let P be a predicate (concept or property), if for any object x, either x completely satisfies P, or completely does not satisfy P, i.e. there does not exist

*Received Nov. 18, 1986.

such object which partially satisfies P and partially does not satisfy P, then P is said to be a distinct predicate and denoted as disp. If for predicate P, there exists such object x which partially has property p and partially does not have property p, then p is said to be a fuzzy predicate, and denoted as fuzp, where dis and fuz represent "distinct" and "fuzzy "respectively. We call the formal symbol $\sim$ fuzzy negative word and read it as " partially " . Therefore, $\sim p(x)$ means object x partially has property p, and $p(x)$ means object x completely has property p.

It was pointed out in [1] that "Since Aristotle, formal logic has made the difference between the inverse opposite and contradictory opposite. If two concepts have their own affirmative contents and have the greatest difference when they are both included in a higher level concept with the same intension, then these two concepts are inverse opposite concepts, for example good and evit, beautiful and ugly etc., If for two concepts, One's intension negates the another's, then these two concepts are contradictory opposite concepts, for example, labour and non-labour, capital and non-capital etc." In fact, the concepts of inverse opposite are everywhere and are frequently used in daily life, social science and natural science, including mathematics. We denote the inverse opposite (negation) as formal symbol ╕ (read: opposite), and denote the inverse opposite side of a predicate p as p. Therefore, p and ╕p are used to represent a pair of the inverse opposite concepts and p and $\neg p$ are used to represent a pair of the contradictory opposite concets. It is well known that in the classical logic, the name of formal symbol $\neg$ is negation and is explained and read as "not". Hence, we use $\neg p$ to represent the contradictory opposit side of a predicate p, and $\neg p(x)$ to represent "not object x completely has property p. " Obviously, $\neg p(x)$ can be derived from ╕$p(x)$.

Given p and ╕p, if a object x satisfies $\sim p(x) \& \sim$ ╕$p(x)$ i.e. x partially has property p and at the same time it partially has property ╕p, then x is said to be the medium object between p and ╕p. In philosophy it is usually said to be "both this-and that" where "this" and "that" namely mean p and ╕p respectively, and "both this-and that" means the medium state in their transformation process of two opposite sides, i.e. the concentrated presentation of inentity in the course of qualitative change, "[2] it looks either like this side or that side of two opposits sides. For example, the dawn is a medium state through which the night changes into the daylight, and the dusk is also a medium state through the daylight changes into the night. They are "both this-and that" states which look like both the daylight and the night. This medium concept of opposite sides is usually used everywhere, from daily life to every area of natural science and social science, for example, the middle-age are both teen-age and old-age, zero is a neutral number which can be regarded as both positive and negative, semiconductor is the intermediate material between conduct and insulation, etc..

§ 2 Non Medium principle and outline and Background of Medium logic System ML Under Medium Principle

In the theory of knowledge, there is a principle which confirms the existence of medium odjects between two opposite sides, where the opposite side are, in fact, always inverse opposite concepts p and $\dashv$p. However, in classical two valued logic and precise classical mathematics, the fuzzy properties and fuzzy concepts, which exist everywhere and are frequently used by people, are excluded from the domain of their research; furthermore, by some restriction of the discourse, the existence of medium objects is denied, the inverse opposite and contradictory opposite are regarded as the same so that $\neg$p is $\dashv$p, i.e. not beautiful is ugly, not good is evil, not true is false etc..

That is to say, in classical logic calculus, the following principle is invisibly carried out: in some appropriate restriction of the discourse, for any predicate p and object x, there is either p(x) or $\dashv$p(x), i.e. without preconditions, for any predicate p there is no x so that $\sim$p(x). We call this principle "non medium principle". It must be pointed out that the classical mathematics did not clearly list non-medium principle as an axiom, while always adhered to this principle in system establishment and development.

Mathematics is a kind of science which studies objective world from quantity aspect and takes puantitative objects as its research objects. However, in every historical period, due to the historical limitation there are always some quantitative objects which are not considered and researched by any way in mathematics. For example, even if mathematics has the long history over 2000 years it was just after 1930s that people truely made use of strict mathematical methods to deal with random quantitative objects. The medium objects, which are excluded from precise classical mathematical system, are just a kind of fuzzy quantitative objects existing in the real world, Nevertheless, prevailing fuzzy mathematics directly use precise classical mathematical methods to deal with this kind of fuzzy quantitative objects. Of course, this is reasonable as people deal with indefinite random phenomena by using definite classical mathematical methods. We have already discussed this in detail in reference[1]. However, is there any non-classical mathematical method which can be used to deal with this kind of fuzzy quantitative objects? Of course, there firstly must be a mathematical system which recognizes the existence of this kind of fuzzy quantitative objects. Hence, we propose to build a set of logic calculas system ML and axiomatic set theory system MS…so called medium mathematical system MM, which recognizes the existence of the medium objects and contains a principle oppositing to the non-medium prineiple. That is to say, we unconditionally recognize that for any predicate p and object x it is not always true that either there exists $p(x)$ or $\neg p(x)$, i.e. there exist such p and x which make both $p(x)$ and $\neg p(x)$ partially true. We call this principle "medium principle." Like classical mathematics, in MM we don't clearly list this medium principle as an axiom either while adhere to this principle in MM (**ML&MS**).

To do so, we must not only directly introduce opposite negative word and fuzzy negative word~, but also set up a series of non-classical logic inference rules. We would like to explain the source and background of such non-classical inference rules in ML by building the insertion law of implication $(\rightarrow_+)$ and the cancellation law of implication $(\rightarrow_-)$ in ML as an example.

It is known that is classical two valued logic, $p \vdash q$, holds $\vdash p \rightarrow q$, but in MP, $p \vdash q$ does not hold $\vdash p \rightarrow q$, because

(*) When p is false or q is trur, $p \rightarrow q$ must be true.

Now for p, following cases may happen:

(1) When p is true, $p \vdash q$ holds q is true, from (*) we know $p \rightarrow q$ is true.

(2) When p is false, from (*) we know $p \rightarrow q$ is true.

(3) When p is medium value~, we do not know whether p is false or q is true. This is why we can not assure that $p \rightarrow q$ is definitely true.

Just because of above case (3), we must set $\sim p \vdash q$ in order that when p takes~value, q must be true, and then from (*) $p \rightarrow q$ is true. That is to say, in MP there must be "$p \vdash q$ and $\sim p \vdash q$" so that we can assure $\vdash p \rightarrow q$. Therefore, $(\rightarrow_+)$ and $(\rightarrow_-)$ in MP will have following form:

$(\rightarrow_+)$ if Γ, $A \vdash B$ and Γ, $\sim A \vdash B$ then $\Gamma \vdash A \rightarrow B$ $(\rightarrow_-)$ $A \rightarrow B$, $A \vdash B$

$A — B$, $\sim A \vdash B$.

It must also be pointed out that in MP the following trichotomy law can be proved: $\vdash A \vee \sim A \vee \neg\!\!\neg A$.

Then, according to the definition $D(\neg)$: $\neg p =_{df} \neg\!\!\neg p \vee \sim p$, we have $p \vee \neg p$.

This indicates that the excluded middle law is still valid in ML, but its content is different from that in classical two-valued logic, i.e. $\neg p$ is no longer regarded as a whole, while it can be divided into $\sim p$ and $\neg p$. Hence, in order to express this difference, above excluded middle law in ML is specially known as sub-excluded middle law. Besides, in classical logic there is the law of contradiction $\vdash \neg(p \vee \neg p)$. However, in ML the followscan be proved:

$\vdash \neg(\text{╕} A \wedge \sim A)$, $\vdash \neg(A \wedge \sim A)$, $\vdash \neg(A \wedge \text{╕} A)$.

Thirdly, in classical logic if $p \vdash q_1, \neg q_1$ and $\neg p \vdash q_2, \neg q_2$, then paradox appears. So does in ML, but t here is a difference between classical logic and ML in the content of $\neg p$. Especially, it must be pointed out that in ML just from $p \vdash q_1, \neg q_1$ and $p \vdash q_2, \neg q_2$, the paradox does not necessarily appear. Because in this case. we have $\vdash \sim p$.

As a particular example, $p \vdash\!\!\dashv \neg p$ is not a paradox in ML when p is a fuzzy predicate, though the reality of paradox $p \vdash\!\!\dashv \neg p$ in classical naive set theory is just $p \vdash\!\!\dashv \text{╕} p$. Then, what means paradox in ML when p is a fuzzy predicate? It must be:

$p \vdash {}_1, \neg {}_1$ and $\text{╕} p \vdash {}_2, \neg {}_2$ and $\sim p \vdash {}_3$.

Lastly, we would like to point out that the ML is quite different from the 3-valued or any multiple-valued logic systems. The differences are at least as follows:

(1) The development of any multiple-valued logic systems always takes classical two-valued logic calculus, which outside the system, as its metalogic tool, while the metalogic idea of ML is medium principle. However, as the medium principle does not confirm that there must always exists medium-object for any oppositive predicate, the metalogic tool of ML does not absolutely exclude the ideal principle of valued logic for any predicate.

(2) As the structure of a logic system depends on its symbol system, rules of formation and rules of inference, i.e. its axiomatic system, one can compare each of the medium logic calculuses ML (MP, MP*, MF, MF*, ME*), which is constracted in the succeeding papers, with a variety of multiple logic systems and finds that these two systems are completely different.

Of course, above (2) only shows the fact that the medium logic calculus ML is different from various multiple logic systems appeared in history, but (1) explains in essence that ML can not be brought into any multiple logic system, i.e. medium logic calculus is substantively different from multiple logic system.

§ 3 The Content and Implication of Cantor's Comprehension Principle

It was well known that Cantor, the founder of set theory, only stated his theory with naive form, there were neither clear primary concepts nor undoubted axiomes. However, abstracting from the classical set theory we can easily find out that the implicit principles of Cantor's theory are just comprehension principle, axiom of extensionality, one-to-one correspondence, principle of infinite extension, principle of relatively complete exhaustion and diagonal process (for detail, see reference [3] Chapter 5, § 2) among which the most important and most influential on mathematics is comprehension principle. Commonly spenking, comprehension principle means that for any predicate (property) p, we can construct a set by collecting all objects and only those objects which satisfy p. This can be expressed by following symbols

$$G=\{g \mid p(g)\} \text{ or } \forall g(g \in G \leftrightarrow p(g))$$

For instance,

$N=\{n \mid n$ is natural number$\}$ or $\forall n(n \in N \leftrightarrow n$ is natural number$)$. In early Cantor's works, the comprehension principle was implicitly applied, while from Frege it cameinto the open. (see: [4]) Briefly speaking, the comprehension principle is the existence axiom (axtom schema) of the following sets: given a formula just containing one free variable x, there is always a set A, which is formed by all x satisfying F.

By analysing above axiom, we can sum up the content and implication of the comprehension principle three nutshells:

Frist, the formula F which is used to form the set must be distinct, i.e. for any object a, $F(a)$ is true or false, and is not partially true or partially false.

Second, there is no restriction to the formula F except that F is distinct, neither is the object universe (i.e. univers of discourse).

Third, for any distinct formula F, the formed set consist of and only of all x satisfying F, i.e. all elements in the set just satisfy F and any object outside the set just does not satisfy F.

It must be pointed out that the frist intension was, in fact, a general premise, by which the comprehension principle was used to form a set in classical s t th ory, and it is also a general premise for constructing set in all axiomatic set theories, which are founded on classical two valued logic calculus. The essence of this premise is refusing to consider and tackle with any fuzzy phenomenon. This means that predicates or formulas which can be used to form sets are all distinct, and there is not any fuzzy predicate which can be used to form set. We call this general premise "not fuzzy set-forming predicate principle." It must be noticed that Cantor and Frege did not clearly point out this essence not only in the description of comprehension principle but also in other cases, and for varieties of modern axiomatic set theories, like BG or ZFC etc., no one definitely state this principle or premise and what they did was gaving tacit consent to this point and adhering to this principle in their mathematical systems. For the second intension, it is confirmed that in the sense of the first intension the selection of set-forming predicates and the universe of discourse is completely free. There is no other restriction. As to the third intension, it is confirmed that there is a strictly one-to-one relationship between set-forming predicates and sets, i.e. any set-forming predicate is uniquely corresponding to a set. Hence, for the convenience of uses and descriptions, people used to replace predicates by their corresponding sets, furthermore, regarded predicates (formula, properly, or concept) and the corresponding sets formed by all objects satisfying the predicates as the same thing. As a matter of fact, this is just a custom for the convenience of uses and description in the sense of the third intension. However, we must emphasize the difference between a predicate and a set. Because the predicate is satisfied by objects, and as soon as the objects which satisfy a predicate form a set, the set does become a object which can have this or that property, but object is not property or predicate by which the object is formed. Therefore we never confuse the two things.

§ 4 The Pan-Comprehension Principle in the Medium Axiomatic Set Theory MS

In order to give a reasonable basis for mathematically dealing with fuzzy phenomena,, e need to build such a mathematical system which accepts fuzzy set-forming predicates and medium objects. First, we must abolish previous restriction denoted as non fuzzy set-forming predicate principle. For this reason, on the basis of the concepts of comprehension set and exact set. We present the pancomprehension principle as follows:

Definition 1 For a predicate p, if set A satisfies conditions:

(1) $p(x) \vdash x \in A$, (2) $\dashv p(x) \vdash x \notin A$,

then A is called comprehension set of p and denoted as Acomp.

Definition 2 For a predicate p, if set A satisfies conditions:

(1) $p(x) \vdash\dashv x \in A$, (2) $\sim p(x) \vdash\dashv x \widetilde{\in} A$, (3) $\dashv p(x) \vdash\dashv x \notin A$,

then A is called exact set of p and denoted as Aexap.

Here $x \widetilde{\in} A$ reads as x partially belongs to A. Of course, in precise classical mathematics, for any object x and set A, either $x \in A$ or $x \notin A$. But here due to the introduction of the opposite negative word $\dashv$ and fuzzy negative word $\sim$, the relation between objects and sets is also correspondingly extended. In particular, for a set A, if there is no x which makes $x \widetilde{\in} A$, then A is called a distinct set denoted as disA. Otherwise, for a set A, if there is a x which makes $x \widetilde{\in} A$, then A is called a fuzzy set and denoted as fuzA.

Obviously, for any AexaP. A is uniquely determined, but for AcomP, A is not uniquely determined.However, the exact set of a predicate p must be a comprehension set of the predicate p, in particular, when predicate p is disP, the comprehension set of p is equal to its exact set and uniquely determined.

Let p be a fuzzy predicate, A, B, C as shown in Figure (4.1) are all comprehension set of p, while A and C are distinct sets, B is a fuzzy set, because there are three X, which make ~p(x), partially belong to B. In Figure (4.2), set D is just the uniquely determined exact set of p. of course, D is also a comprehension set of p. It must be noticed that for any X which makes X $\tilde{\in}$ D, we do not give the degree values at which X partially belong to D. In Figure (4.3), when p is distinct predicate, set E is the sole exact set of p, and is also the sole comprehension set of p, i.e. the comprehension set of p is equal to its exact set, and uniquely de ermined in this case .

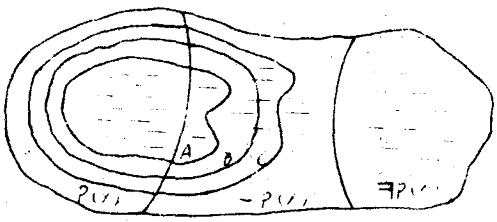

Figure (4.1)

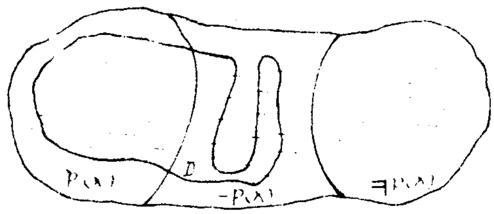

Figure (4.2)

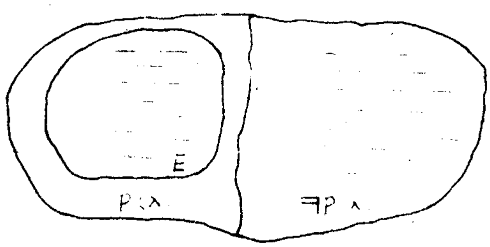

Figure (4.3)

Now we introduce the following axiom in MS:

Pan-Comprehension Principle For a predicate p, no matter it is dis or fuz, there is at least a set A which satisfies AcomP.

It can be seen after analysis that on the one hand, there are some important differences between comprehension principle and pan-comprehension principle, first, the restriction of non-fuzzy set-forming predicate in comprehension principle is abolished, and secondly in AcomP, A may consist of not only all x which make p(x), but also some x which make ~p(x), thirdly an comprehension principle differs from the third intension of the comprehension principle, because in AcomP there is no strict one-to-one relationship between the set A and the predicate , i.e. the comprehension set of a predicate is not uniquely determined. On the other hand, in a sense, comprehension principle is also fully reserved in pan-comprehension principle because when we only research the precise phenomenon, not the fuzzy phenomenon, of the real world, for any disp there always is a uniquely determined Aexap. That is to say that in MS the Contor's comprehension principle can be expressed as: for any disP, the comprehension set of is equal to its exact set, and uniquely determined, it consist of and only of all x satisfying (x).

Perhaps one worries about that paradoxes must be appeared in MS. In fact it is not true. We can effectively exclude any historical logic-mathematical paradoxes in MS, because the logic calculus which matches the medium axiomatic set theory MS is not the classical two valued logic calculus but the medium logic system ML. For Russell paradox, we can explain it as follows.

For a formula $x \notin x$, according to pan-comprehension principle we have Acom$(x \notin x)$, i.e. set A satisfies the condition: $x \notin x \vdash\!\!- x \in A$ and $x \in x \vdash\!\!- x \notin A$. Instead of x with A, we have $A \notin A \dashv\vdash A \in A$. However in ML this is not a paradox, but only indicate that we have $\vdash\!\!- \sim (x \notin x)$, i.e. $x \in x$ is a fuzp. Furthermore, Acom$(x \in x)$ is a fuzA, and A itself is the medium object between $x \notin x$ and $x \in x$.

§ 5 Purpose and Significance of Constructing the Medium Mathematical System MM (ML & MS)

In order to state the purpose and significance of constructing MM, two problems should be firstly answered: the first is how to solve the consistency problem in classical mathematics; the second is how to establish the theoritical foundation for fuzzy mathematics.

For the first problem, since paradoxes were found in set theory, which is the theoritical foundation of whole classical mathematics, people have widespreadly doubted that the comprehension princple ifself is the absurd cause of ruin, because it was found out that according to the comprehension principle it could be derived that $\Sigma = \{x \mid x \notin x\}$, $E = \{x \mid x$ is a set$\}$ etc. were sets and were not sets. i.e. the comprehension principle led to the contradictory conclusion. One of the views appeared then was to revise the comprehension principle but not to change the classical two valued logic calculus which was matched with set theory. Ramsey's simple type theory, Bernays-Gödel's BG system and the wide used ZFC system are all based on the revision of comprehension principle and obtained a certain success in excluding paradoxes, i.e. all the paradoxes appeared in history are excluded and no new paradox has been found in above systems. The other view appeared was not to revise comprehension principle but to change the classical two valued logic system which was matched with set theory, such as Бочевар's multiple-valued logic system. However, reference [5] has firstly proved that if any mathematical system satisfies conditions: (1) the comprehension principle holds, (2) $p \to p$ (3) $(p \to)^{n+1}q$ inters $(p \to)^n q$, then this system must contain paradoxes. The reference [5] has also proved that by adding comprehension principle, Łukasiewicz's finite valued logic system $Ł'_n (3 \leq n < \omega)$ are just such systems. Nevertheless, the method given by reference [5] can not be used to decide whether or not Łukasiewicz's continuous valued logic system is consistent with comprehension principle, because the system does not satisfy above condition (3). In 50's and 60's of this century, above problem came to some preliminary results. Skolem, C. C. Chang [6] and J. E. Fenstad [7] proved respectively the consistency of Łukasiewicz's continuous value logic system Ł (or $Ł_i$) with several particular types of sets of comprehension principle formula (*) such as $\Sigma_1, \Sigma_2, \Sigma_3$, but as to the consistency of Ł or $Ł_i$ with the sets Σ_0 of all comprehension principle formula there was still no any clear answer. It must be pointed out that even if the positive answer can be reached, it does not indicate that the comprehension principle plus axioms of other set theories can still be consistent with Ł or $Ł_i$. For example, C. C. Chang pointed out that Σ_2, Σ_3 plus the axiom of extensionality is no longer consistent with Ł. Obviously, use other axioms of set theories to develop mathematics.

In order to develop the whole mathematics in the continuous valued logic system and reserve comprehension principle at the same time one must prove the consistency of the continuous valued logic system with a set theory axiomatic system which is rich enough in intension and includes the comprehension principle. Can this aim be achieved? We and our cooperators answered this question negatively in the reference [8] in which it was proved that if any mathematical systom satisfies the conditions: (1) the comprehension principle holds, (2) $p,\ p \to q \vdash q$, (3) $p \to p$, (4) infinite union operation of sets is allowed, (5) natural number system is contained, then this system definitely includes paradoxes. Because any mathematical system which is rich enough in intension always satisfies condition (2) … (5) it is obvious that one can not exclude paradoxes just by changing logic system without revising the compreh-

(*) In the first order logic, the comprehension principle is not a axiom but a axiom schema, i.e. infinite axioms: $\forall x_1 \cdots \forall x_n \exists y \forall t\ (t \in y \leftrightarrow \varphi(t, x_1, \cdots, x_n))$. Giving different restrictions to formula φ, we have different sets of comprehension principle formula such as Σ_1, Σ_2, etc.

ension principle.

Hence, the comprehension principle must be revised. But the problem is how we should revise comprehension principle so as to reserve its reasonable content to a maximum extent. However all historical revision programmes were not satisfactory.

For the second problem, fuzzy mathematics is a new discipline which was developed just from 1960s. Not only is there no a firm theoritical foundation for it, but also that no what kind of theoritied basis it can be laid is still a question to research. At present, there may be three programmes, first, fuzzy mathematics is directly or indirectly laid on the basis of ZFC axiomatic set theory. However the so developed fuzzy mathematics would just be a branch of the whole classical mathematics, and the tools and methods used to deal with fuzzy phenomena would be restricted in the extent of classical mathematical methods. For example, Zadeh's and Gentihomme's programmes are such one's.[9]、[10]、[11] In fact, the initial definitions of fuzzy subset and the method of constructing a fuzzy set presented by Zadeh have already determined that such a discipline must be based on classical two valued logic calculus and ZFC axiomatic set theory. Manes's idea of defining the mathematical structure (T, e, $(-)^{\#}$) in the classical mathematical frame which is called as generalized fuzzy theory by Manes, is also a programme which directly base fuzzy mathematics on ZFC.[12] Wang peizhuong's research of "fuzzy set and stochastic set shadow" is a method by which a medium theoretical link frame is built between the ZFC axiom set theory and the fuzzy mathematics, then fuzzy mathematics is indirectly based on the ZFC system. The second possible programme is to build a particular axiom system of fuzzy mathematics. For example, the "set-valued set theory" developed by Chapin and Weitner in references [14]—[16] is such a system and is independent of ZFC system. It consists of 9 axioms and is briefly called ZB system. However, they still used the first order theory of the classical two valued logic as its metalogic, because they had not developed the particular logic calculus as a formal language using to describe ZB system for fuzzy mthematics. The third programme is to build a set of fuzzy mathematics which is fully independent of the classical mathematics. It can be based on the acception of fuzzy set-forming predicates and medium objects so that this kind of fuzzy mathematies can have its own logic calculus system and axiomatic set theory system, and therefore particular research methods. However, so built fuzzy mathematics not only should not exclude absolutely precise classical mathematics but also should include precise classical mathematics in a higher form.

Therefore the purpose and significance of constructing MM(ML & MS) can be summarized as follows:

(1) In the view of medium principle, we introduced the fuzzy phenomena, which were excluded by the classical mathematics but usually used everywhere, into MM, therefore the MM greatly extends the domain of objects of mathematics.

(2) Various historical programmes which revised comprehension principle were all based on the revision of its second intension and excessively excluded the reasonable content of the comprehension principle as the cost of excluding the paradoxes. However, the pan-comprehension principle in MM is based on the full revision of the third intension of the comprehension principle, and also reserves the reasonable content of comprehension principle in a maximum extent in the condition of not regarding the fuzzy predicates.

(3) Various historical programmes which exclude paradoxes either excessively excluded the reasonable content of comprehension principle when revised it, or were based on changing logic system thus led to failure[8]. MM both changes the logic system and revises the comprehension principle, which not only reserves the reasonable content of comprehension principle in a maximum extent but also gives a reasonable explanation for all known logic mathematical paradoxes.

(4) Because of the introduction of the fuzzy set forming predicates and the acception of the medium objects in MM, there exist more generalized concepts and principles which can be used to directly explain various medium objects and fuzzy concepts which are "both this and that". Therefore the MM puts forward a fully reasonable and allround programme which will be a theoritical foundation of fuzzy mathematics, and thus will provide a particular logic system ML and set theory system MS for the research of fuzzy phenomena.

(5) MM greatly extends the logic foundation of classical mathematics and the foundation of set throry, i.e. MM extends the classical two valued logic into medium logic ML, and the ZFC axiomatic set theory into medium axiomatic set theory MS. Therefore, the MM provides a possibility for constructing the fuzzy mathematics which includes the precise classical mathematics. Of course, the theoretical content and research methods of this fuzzy mathematics are completely different from present fuzzy mathematics which is directly based on ZFC system.

(6) Because any formula in ML can be clarified by the clarifying operator of ML, when we needn't to consider the fuzzy phenomena, we can reduce the MS and ML into ZFC system and classical two valued logic calculus respective-

ly with the clarifying operator. That just means that the precise classical mathematics is included in a higher form in the medium mathematical system MM.

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The author is indebted to Prof. Z. Q. Xia for his helpful suggestions and comments.

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