J.Math.Res.& Exposition, 1988, No. 1

On the Combinatorial Compound Matrix*

Richard A. Brualai[1]
University of Wisconsin
Madison. U. S. A.

Li Qiao (李 乔)
University of Science and
Technology of China

Abstract

We define the r^{th} combinatorial compound $C_r^*(A)$ of a matrix A, which can be viewed as the characteristic function of the subset of the $r\times r$ submatrices of A which are combinatorially nonsingular. We prove that for $1\leqslant r<n$, A is fully indecomposable if and only if $C_r^*(A)$ is. We determine the minimum number of 2×2 and 3×3 combinatorially nonsingular submatrices over all $n\times n$ fully indecomposable matrices and make a conjecture for general r.

I. Introduction

Let $A=[a_{ij}]$ be an $n\times n$ matrix, and let r be an integer with $1\leqslant r\leqslant n$. Let $Q_{r,n}$ be the collection of all strictly increasing sequences of length r taken from $\{1,2,\cdots,n\}$. Thus $(i_1,i_2,\cdots,i_r)\in Q_{r,n}$ if and only if $1\leqslant i_1<i_2<\cdots<i_r\leqslant n$. For $\alpha,\beta\in Q_{r,n}$, $A[\alpha|\beta]$ denotes the $r\times r$ submatrix of A whose rows are indexed by the terms of α and whose columns are indexed by the terms of β. The classical r^{th} compound of A, denoted by $C_r(A)$, is defined as follows. Let the members of $Q_{r,n}$ be arranged in lexicographic order. Then $C_r(A)$ is the $\binom{n}{r}\times\binom{n}{r}$ matrix whose rows and columns are indexed by $Q_{r,n}$ and whose (α,β)-entry is

$$c_{\alpha,\beta}=\det[A[\alpha|\beta] \qquad (\alpha,\beta\in Q_{r,n}).$$

The r^{th} compound has some interesting properties [6,8] and has been useful in certain combinatorial investigations.

We define here a combinatorial version of compound matrices and investigate some of its properties. A set of entries of the matrix A is called independent if no two of them come from the same row or column. The term rank of A, denoted by $\rho(A)$, is the maximum cardinality of an independent set of nonzero entries. It is well known [9] that $\rho(A)$ is the minimum number of rows and columns of A which contain all its nonzero entries. We define A to be combinatorially nonsingular if $\rho(A)=n$. One reason for this definition is that when the

* Received Feb. 21, 1984.
1) Recearch partially supported by a National Science Foundation Grant .

nonzero entries of A are replaced by distinct, algebraically independent indeterminants, the resulting matrix is nonsingular if and only if A is combinatorially nonsingular. The r^{th} combinatoria compound $C_r^*(A)$ of A is the $\binom{n}{r}\times\binom{n}{r}$ matrix whose rows and columns are indexed by $Q_{r,n}$ and whose (α,β)-entry $c^*_{\alpha,\beta}$ satisfies $c^*_{\alpha,\beta}=\begin{cases}1, & \text{if } A[\alpha|\beta] \text{ is combinatorially nonsingular.}\\ 0, & \text{otherwise.}\end{cases}$ The matrix $C_r^*(A)$ is a matrix of $0'$s and $1'$s, and there is no loss of generality in assuming from now on that A is also a matrix of $0'$s and $1'$s. It then follows that $C_1^*(A)==A$ and that $C_n^*(A)=[1]$ or $[0]$ according as to whether A is combinatorially nonsingular or combinatorially singular. In addition, if $A=J_n$, the $n\times n$ matrix of all $1'$s, then $C_r^*(J_n)=J_{\binom{n}{r}}$ for each r with $1\leqslant r\leqslant n$.

Recall that an $n\times n$ matrix $A=[a_{ij}]$ is called reducible if there exists a permutation matrix P such that PAP' has the form

$$(1)\qquad \begin{bmatrix} A_1 & O \\ X & A_2 \end{bmatrix},\ A_1 \text{ and } A_2 \text{ are square and nonvacuous.}$$

The matrix A is irreducible if it is not reducible. It is well known that A is irreducible if and only if its associated directed graph D_A is strongly connected [10]. The vertices of D_A are $1,2,\cdots,n$ and there is an arc from i to j if and only if $a_{ij}\neq 0$. D_A is strongly connected means for each ordered pair of vertices k, l there is a directed path from k to l. The matrix A is partly decomposable if either $n=1$ and $A=[0]$, or $n>1$ and there are permutation matrices P and Q such that PAQ has the form (1). The matrix A is fully indecomposable if it is not partly decomposable. If the matrix A has all $1'$s on its main diagonal, then A is irreducible if and only if A is fully indecomposable [2]. Fully indecomposable matrices arise in several different settings, for instance in the study of doubly stochastic matrices and permanents [3,7]. It is well known that A is fully indecomposable if and only if every $(n-1)\times(n-1)$ submatrix of A has term rank equal to $n-1$ [1]. In particular, for A fully indecomposable every 1 belongs to a set of n independent $1'$s. In the language above we conclude that A is fully indecomposable if and only if $C^*_{n-1}(A)=J_n$. We prove here that for $1\leqslant r\leqslant n$, $C_r^*(A)$ is fully indecomposable if and only if A is.

A matrix A is permutation equivalent to a matrix B if there are permutation matrices R and S with $A=RBS$. For $n>1$, let $P_n=[p_{ij}]$ denote the $n\times n$ matrix of $0'$s and $1'$s where $p_{12}=p_{23}=\cdots=p_{n-1,n}=p_{n,1}=1$ and $p_{ij}=0$, otherwise. We let $F_n=I_n+P_n$, where I_n is the $n\times n$ identity matrix. The matrix F_n is readily seen to be fully indecomposable. For A an $n\times n$ matrix of $0'$s and $1'$s, let $\sigma(A)$ equal the number of $1'$s of A. Since an $n\times n$ fully indecomposable matrix with $n>1$ has at least 2 $1'$s in each row and column, each $n\times n$ fully indecomposable matrix A with $n>1$ satisfies $\sigma(A)\geqslant 2n$. It is easily checked that equality holds if anh only if A is permutation equivalent to F_n. We conjecture here that for an $n\times n$ fully indecomposable matrix A with $n>1$ and for $1<r<n$

(2) $$\sigma(C_r^*(A))\geq\sigma(C_r^*(F_n)).$$

That is, F_n has the smallest number of combinatorially nonsingular $r\times r$ submatrices among all $n\times n$ fully indecomposable matrices. When $r=1$, the inequality (2) holds by the above comment and equality is attained if and only if $RAS=F_n$ for some permutation matrices R and S. For $r=n$, $C_n^*(A)=J_1$ and (2) is an identity. For $r=n-1$, $C_{n-1}^*(A)=J_n$ and (2) is again an identity. For A nearly decomposable (see Section 3), we conjecture that for $2\leq r\leq n-2$, equality holds only when A is permutation equivalent to F_n. We prove both conjectures for $r==2$ and 3.

2. Full indecomposability

In this section we prove that the property of full indecomposability is inherited from a matrix by its r^{th} combinatorial compound. We begin by giving the definition for the directed graph analogue of the r^{th} combinatorial compound.

Let D be a directed graph with vertex set $V(D)=\{1,2,\cdots,n\}$, and let r be an integer with $1\leq r\leq n$. The r^{th} compound of D, denoted by D_r^*, is defined as follows. The set $V(D_r^*)$ of vertices of D_r^* is the set $Q_{r,n}$ of strictly increasing sequences of length r taken from $\{1,2,\cdots,n\}$. Let $\alpha=(i_1,\cdots,i_r)$ and $\beta=(j_1,\cdots,j_r)$ be in $Q_{r,n}$. Then there is an arc in D_r^* from α to β if and only if

(3) For some permutation $k_1,\cdots,k_r$ of $\{1,\cdots,r\}$, there is an arc in D from i_t to j_{k_t} for $t=1,\cdots,r$.

From the definitions of the r^{th} combinatorial compound of a matrix and the r^{th} compound of a directed graph, it follows that for A an $n\times n$ matrix of 0's and 1's,

(4) $$D_{C_r^*(A)}=(D_A)_r^*.$$

Recall that a loop at avertex v of a directed graph is an arc from v to itself. We first prove the following.

Theorem 1 Let D be a directed graph with vertex set $V(D)=\{1,2,\cdots,n\}$ such that D has a loop at each vertex. Let r be an integer with $1\leq r\leq n$. If D is strongly connected, then D_r^* is also strongly connected and D_r^* has a loop at each vertex.

Proof Clearly, D_r^* has a loop at each vertex. Suppose D is strongly connected. We need to show that for each ordered pair α,β of distinct vertices of D_r^*, there is a directed path from α to β. For the remainder of this proof we regard members of $Q_{r,n}$ as subsets of r elements of $\{1,2,\cdots,n\}$, which can be then arranged in strictly increasing order. Let $\alpha=\{i_1,\cdots,i_s,k_1,\cdots,k_t\}$ and $\beta=\{j_1,\cdots,j_s,k_1,\cdots,k_t\}$ where $s+t=r$, $s\geq1$, and $\alpha\cap\beta=\{k_1,\cdots,k_t\}$. Let $X=\{i_1,\cdots,i_s\}$, $Y=\{j_1,\cdots,j_s\}$, and $Z=\{k_1,\cdots,k_t\}$. Since D is strongly connected, there is a directed path P in D from a vertex in X to a vertex in Y. We choose P to be such a path of minimum length and may assume i_s is its first vertex and j_s is its last vertex. In particular, P is a simple path meeting X only at i_s and meeting Y only at j_s. It suffices to prove the following:

(∗) There is a directed path in D_r^* from a to some vertex y such that $|y \cap \beta| > t$.

We prove (∗) by induction on the number m of vertices of P which belong to Z. First suppose $m=0$. Then since D has a loop at each vertex, we easily construct from P a directed path of D_r^* from a to $y=(a-\{i_s\})\cup\{j_s\}$ where $y\cap\beta = Z\cup\{j_s\}$. Hence $|y\cap\beta|>t$, and (∗) holds in this case. Now let $m\geqslant 1$. Let the first arc of P which enters Z be (i_s', k_1) and let the first arc of P which leaves Z be (k_p, j_s') (see Figure 1). Let that part of the path P from k_1 to k_p be $P(k_1, k_2, \cdots, k_p)$

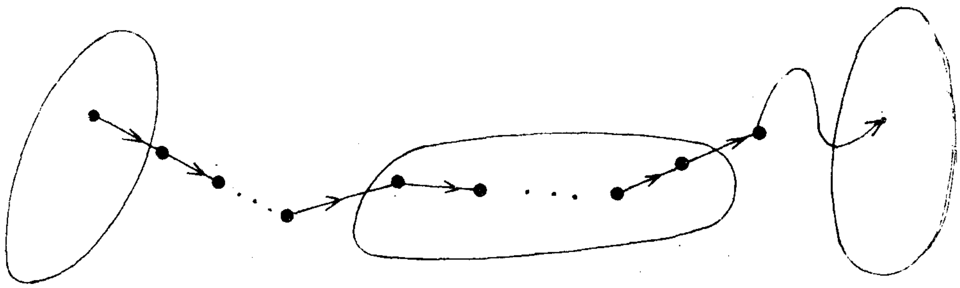

Figure 1: The path P in D

Since D has a loop at each vertex, there is a directed path of D_r^* from $a=\{i_1, \cdots, i_{s-1}, i_s, k_1, \cdots, k_t\}$ through $a_0=\{i_1, \cdots, i_{s-1}, i_s', k_1, \cdots, k_t\}$ to $a_1=\{i_1, \cdots, i_{s-1}, k_1, \cdots, k_p, j_s', k_{p+1}, \cdots, k_t\}$. If $j_s' \in Y$, then in (∗) we may take $y=a_0$ and we have a path in D_r^* from a to y with $|y\cap\beta|=t+1$. Now suppose $j_s' \notin Y$. Then the part of P from j_s' to j_s, $P(j_s', \cdots, j_s)$, is a path of D of minimum length from a vertex in $X'=\{i_1, \cdots, i_{s-1}, j_s'\}$ to Y. Since $a_1\cap\beta = Z$ and the path $P(j_s', \cdots, j_s)$ has $m-p<m$ vertices in common with Z, it follows by the inductive assumption that there is a directed path in D_r^* from a_1 to some vertex y such that $|y\cap\beta|>t$. Since there is a directed path in D_r^* from a to a_1, there is a directed path from a to y where $|y\cap\beta|>t$. Hence (∗) holds by induction, and the proof of the theorem is complete.

Theorem 2 Let A be an $n\times n$ matrix of 0's and 1's, and let r be a fixed integer with $1<r<n-1$. Then A is fully indecomposable if and only if $C_r^*(A)$ is fully indecomposable.

Proof First suppose A is fully indecomposable. Since every $(n-1)\times(n-1)$

submatrix of A has term rank $n-1$, there is a permutation matrix $P<A$. Let B be a matrix obtained from A by row and column permutations. Then B is also fully indecomposable and $C_r^*(B)$ can be obtained from $C_r^*(A)$ by row and column permutations. Hence there is no loss of generality in assuming $I_n<A$. But then A is irreducible, and D_A is a strongly connected directed graph with a loop at each vertex. By Theorem 1, $(D_A)_r^*$ is strongly connected with a loop at each vertex. Hence by (4) $C_r^*(A)$ is an irreducible matrix with $I_{\binom{n}{r}}<C_r^*(A)$ and thus $C_r^*(A)$ is fully indecomposable.

Now suppose that A is partly decomposable. Without loss of generality we may assume $A=\begin{bmatrix} A_1 & O \\ X & A_2 \end{bmatrix}$ where A_1 is $k\times k$ with $k\geqslant 1$. We may assume $k\leqslant n-k$. We show $C_r^*(A)$ is partly decomposable by determining nonempty $\alpha^*,\beta^*\subseteq Q_{r,n}$ such that $|\alpha^*|+|\beta^*|=\binom{n}{r}$ and $\rho(A[\alpha|\beta])<r$ for $\alpha\in\alpha^*$ and $\beta\in\beta^*$, We distinguish two cases.

Case 1: $1\leqslant r\leqslant n-k$.

Let
$$\alpha^*=\{\alpha\in Q_{r,n} | \alpha\cap\{1,\cdots,k\}\neq\phi\}$$
$$\beta^*=\{\beta\in Q_{r,n} | \beta\subseteq\{k+1,\cdots,n\}\}.$$

Then $|\alpha^*|=\binom{n}{r}-\binom{n-k}{r}$ and $|\beta^*|=\binom{n-k}{r}$, so that $|\alpha^*|+|\beta^*|=\binom{n}{r}$. Moreover, since $A[\alpha|\beta]$ has at least one row of all 0's, $\rho(A[\alpha|\beta])<r$ for $\alpha\in\alpha^*$, $\beta\in\beta^*$.

Case 2: $r=n-k+t$, $1\leqslant t\leqslant k-1$.

Let
$$\alpha^*=\{\alpha\in Q_{r,n} | \,|\alpha\cap\{1,\cdots,k\}|\geqslant t+1\}$$
$$\beta^*=\{\beta\in Q_{r,n} | \{k+1,\cdots,n\}\subseteq\beta\}.$$

Then $|\alpha^*|=\sum_{i=1}^{k-t}\binom{k}{t+i}\binom{n-k}{r-(t+i)}$ and $|\beta^*|=\binom{k}{t}$. Hence
$$|\alpha^*|+|\beta^*|=\sum_{i=0}^{k-t}\binom{k}{t+i}\binom{n-k}{r-(t+i)}=\sum_{j=t}^{k}\binom{k}{j}\binom{n-k}{r-j}=\binom{n}{r}.$$

Moreover, for $\alpha\in\alpha^*,\beta\in\beta^*$, $A[\alpha|\beta]$ has a zero submatrix of size $(t+1)\times(n-k)$ where $t+1+(n-k)=r+1$ and hence $\rho(A[\alpha|\beta])<r$. This completes the proof of the theorem.

Note that for $r=n$, A fully indecomposable implies $C_n^*(A)=[1]$, a fully indecomposable.

3. Proof of the conjectures for $r=2$ and 3

Let A be an $n\times n$ fully indecomposable matrix of 0's and 1's. Then A is called nearly decomposable if each matrix obtained from A by replacing a 1 by 0 is partly decomposable. We shall need the following two properties of nearly decomposable matrices which we state as lemmas.

Lemma 1[5] Let A be an $n\times n$ nearly decomposable matrix of 0's and 1's. If $n>2$, then J_2 is not a submatrix of A.

Lemma 2[4] Let A be an $n\times n$ nearly decomposable matrix. Then there exists an integer s with $1\leqslant s\leqslant n-1$ and an $(n-s)\times(n-s)$ nearly decomposable matrix A' such that A is permutation equivalent to

$$(5)\qquad \left[\begin{array}{cccc|ccc} 1 & & & & & & 1\\ 1 & \ddots & & & & & \\ & \ddots & \ddots & & & & \\ & & 1 & 1 & & & \\ \hline & & & 1 & & & \\ & & & & & A' & \\ & & & & & & \end{array}\right]\begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} s \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} n-s \end{array}$$

Where unspecified entries are $0'$s. If A is not permutation equivalent to F_n, then $s\leqslant n-3$.

In addition we require the following two lemmas which are easily obtained from known results.

Lemma 3 For $1\leqslant r\leqslant n$,

$$\sigma(C_r^*(F_n))=\frac{2n}{2n-r}\binom{2n-r}{r}=\frac{2n}{r}\binom{2n-r-1}{r-1}.$$

Proof The result holds when $r=1$, since $\sigma(C_1^*(F_n))=\sigma(F_n)=2n$. Suppose $2\leqslant r\leqslant n$. Then an $r\times r$ submatrix B of F_n satisfying $\rho(B)=r$ has exactly one independent set of r $1'$s. Hence the number of $r\times r$ submatrices B of F_n with $\rho(B)=r$ is the same as the number of independent sets of r $1'$s of F_n. From the definition of F_n, this is the same as the number $g(2n,r)$ of ways to select r objects, no two consecutive, from $2n$ objects arranged in a circle. By a theorem of Kaplansky (see [9, p. 34]), $g(2n,r)=\frac{2n}{2n-r}\binom{2n-r}{r}$ and the lemma follows.

Lemma 4 Let Q_n be the $n\times n$ matrix

$$\begin{bmatrix} 1 & & & O \\ 1 & 1 & & \\ & \ddots & \ddots & \\ O & & 1 & 1 \end{bmatrix}$$

Then for $1\leqslant r\leqslant n$, $\sigma(C_r^*(Q_n))=\binom{2n-r}{r}$.

Proof $\sigma(C_r^*(P_n))$ is the number $f(2n-1,r)$ of ways to select r objects, no two consecutive, from $2n-1$ objects arranged in a line. By another theorem of Kaplansky (see [9, p. 33]),

$$g(2n,r)=\binom{2n-r}{r}.$$

Theorem 3 Let A be an $n\times n$ fully indecomposable matrix of $0'$s and $1'$s with $n>2$. Then

$$(6)\qquad \sigma(C_2^*(A))\geqslant n(2n-3).$$

If $n=3$, equality holds for all fully indecomposable A. For $n>3$ and A nearly decomposable, equality holds if and only if A is permutation equivalent to F_n.

Proof Clearly it suffices to prove (6) when A is nearly decomposable.

Thus let A be nearly decomposable. By Lemmal 1, J_2 is not a submatrix of A, and hence $\sigma(C_2^*(A)$ equals the number of independent pairs of $1'$s of A. We first show that every 1 of A belongs to at least $2n-3$ independent pairs.

Consider any 1 of A and let B be the matrix obtained by replacing it with a 0. Since A is nearly decomposable, B is partly decomposable and it follows that B is permutation equivalent to a matrix of the form

$$(7)\qquad \begin{bmatrix} B_1 & O & O & \cdots & O \\ B_{21} & B_2 & O & \cdots & O \\ B_{31} & B_{32} & B_3 & \cdots & O \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{t,1} & B_{t,2} & & \cdots & B_t \end{bmatrix} \qquad (t \geqslant 2)$$

where B_i is an $n_i \times n_i$ fully indecomposable matrix. Without loss of generality we may assume B has the form (7) and that A is obtained from B by replacing the $(1,n)$-entry of B with a 1 (the 1 of A that was replaced by 0 to give B). Since B_i is fully indecomposable, when $n_i > 1$, B_i has at least two $1'$s in each row and column and in particular $\sigma(B_i) \geqslant 2n_i$ $(i = 1, \cdots, t)$. Since A is fully indecomposable,

$$\sigma(B_{k1} \cdots B_{k,k-1}) \leqslant 1 \quad (k = 2, \cdots, t).$$

It now follows that the matrix obtained from A (or from (7)) by striking out row 1 and column n has at least $2(n-2)+1 = 2n-3$ $1'$s. Hence for each 1 of A there are at least $2n-3$ other $1'$s lying in a different row and column from the given 1. It follows that

$$(8)\qquad \sigma(C_2^*(A)) \geqslant \frac{\sigma(A)(2n-3)}{2}$$

where we have divided by 2 because some pairs of $1'$s accounted for on the right hand side may be counted twice. Since $\sigma(A) \geqslant 2n$, (6) follows from (8). Suppose equality occurs in (6). Then it follows that $\sigma(A) = 2n$ and hence A is permutation epuivalent to F_n, and it follows from Lemma 3, that $\sigma(C_2^*(F_n)) = n(2n-3)$.

Before considering the case $r=3$, we obtain an additional lemma.

Lemma 5 Let A be an $n \times n$ nearly decomposable matrix of $0'$s and $1'$s where $n \geqslant 4$. Let B be obtained from A by striking out a row or a column. Then

$$\sigma(C_2^*(B)) \geqslant (n-1)^2.$$

Proof Without loss of generality let B be obtained from A by striking out row 1. Since A is nearly decomposable and $n > 2$, J_2 is not a submatrix of B. Thus $\sigma(C_2^*(B))$ is the number of independent pairs of $1'$s of B. Since $n > 1$ A has at least two $1'$s in each row and column. Thus B has at least two $1'$s in each row, and in particular $\sigma(B) \geqslant 2(n-1)$.

First suppose each 1 of the $2(n-1)$ $1'$s of B known to exist belongs to at least $n-1$ pairs of independent $1'$s. Then

$$\sigma(C_2^*(B))\geqslant\frac{2(n-1)(n-1)}{2}=(n-1)^2,$$

where we have divided by 2 since each pair may be counted twice.

Now suppose that some 1 of B (referred to as the given 1) belongs to at most $n-2$ pairs of independent $1'$s. But since A is nearly decomposable, each 1 of A belongs to a set of n independent $1'$s, and hence each 1 of B belongs to a set S of $n-1$ independent $1'$s. This set S already accounts for $n-2$ pairs of idependent $1'$s containing the given 1. Without loss of generality we may assume that

$$B=\left(\begin{array}{c|cccc} a & y_1 & \cdots & y_{n-2} & 1\\ \hline * & & & 1 & x_1\\ \vdots & & & * & \vdots\\ \vdots & & \cdot & & \vdots\\ \vdots & & * & & \vdots\\ \vdots & \cdot & & & \vdots\\ * & 1 & & & x_{n-2}\end{array}\right)$$

where the given 1 is in the upper right hand corner and the $n-1$ displayed $1'$s are those of S. All the asterisked positions must be occupied by $0'$s; otherwise the given 1 belongs to more than $n-2$ pairs of independent $1'$s. Since A has at least two $1'$s in each row and column, $a=x_1=\cdots=x_{n-2}=1$. Hence

$$B=\left(\begin{array}{c|cccc} 1 & y_1 & \cdots & y_{n-2} & 1\\ \hline 0 & & & 1 & 1\\ \vdots & \bigcirc & & & \vdots\\ \vdots & & \cdot & & \vdots\\ \vdots & & & \bigcirc & \vdots\\ 0 & 1 & & & 1\end{array}\right)$$

We now have $2(n-1)$ $1'$s of B displayed. We count the number of pairs of independent $1'$s to which each belongs and divide by 2 to get a lower estimate for the number of pairs of independent $1'$s. We obtain

(1 in upper left): $2(n-2)$ (1 in upper right): $n-2$

($1'$s in lower right): $(n-2)\cdot(n-2)$ (the remaining $1'$s): $(n-2)\cdot(2(n-2))$.

Hence

$$\sigma(C_2^*(B))\geqslant\frac{3(n-2)(n-1)}{2}.$$

But for $n\geqslant4$, $\dfrac{3(n-2)(n-1)}{2}\geqslant(n-1)^2$, and the lemma is proved.

By taking A to be F_3 and B to be the matrix obtained from A by crossing out row 1, we see the lemma is not true for $n=3$: the lower bound is 3 ra ther than the 4 given in the lemma.

Theorem 4 Let A be an $n\times n$ fully indecomposable matrix with $n\geqslant 4$. Then

$$\sigma(C_3^*(A))\geqslant\frac{2n(n-2)(2n-5)}{3}. \tag{9}$$

If $n=4$, equality holds for all fully indecomposable A. For $n>4$ and A nearly decomposable, equality holds if and only if A is permutation epuivalent to F_n.

Proof We prove the theorem by induction $n\geqslant 4$. It suffices to prove (9) when A is nearly decomposable. Let A be nearly decomposable. If $n=4$, then (9) becomes $\sigma(C_3^*(A))\geqslant 16$ which holds with equality, since every 3×3 submatrix B of A satisfies $\rho(B)=3$. Now let $n>4$. If A is permutation equivalent to F_n, then it follows from Lemma 3 that (9) holds with equality. Hence to complete the proof, it suffices to show that when A is not permutation equivalent to F_n, (9) is a strict inequality. We now suppose that A is not permutation equivalent to F_n. By Lemma 2 we may assume A has the form (5) where s is an integer with $1\leqslant s\leqslant n-3$ and A' is nearly decomposable. We estimate the number of 3×3 submatrices $A[\alpha|\beta]$ of A with term rank equal to 3.

(i) $\alpha,\beta\subseteq\{s+1,\cdots,n\}$: By the inductive assumption we get at least

$$\sigma(C_3^*(A'))\geqslant\frac{2(n-s)(n-s-2)(2n-2s-5)}{3}.$$

(ii) $|\alpha\cap\{s+1,\cdots,n\}|=2=|\beta\cap\{s+1,\cdots,n\}|$: By Theorem 3 we get at least

$$(2s-1)\sigma(C_2^*(A'))\geqslant(2s-1)(n-s)(2n-2s-3).$$

(iii) $|\alpha\cap\{s+1,\cdots,n\}|=1=|\beta\cap\{s+1,\cdots,n\}|$: Using Lemma 4 and the fact that a fully indecomposable matrix which is not 1×1 has at least 2 $1'$s in each row, we get

$$\binom{2s-2}{2}2(n-s)=(2s-2)(2s-3)(n-s).$$

(iv) $\alpha,\beta\subseteq\{1,\cdots,s\}$: By Lemma 5 we get at least

$$\binom{2s-3}{3}=\frac{(2s-3)(s-2)(2s-5)}{3}.$$

We now make use of the $1'$s in the upper right and lower left blocks of (5).

(v) $\alpha\cap\{1,\cdots,s\}=\{1\}$, $\beta\cap\{1,\cdots,s\}=\phi$ or
$\alpha\cap\{1,\cdots,s\}=\phi$, $\beta\cap\{1,\cdots,s\}=\{s\}$: Using Lemma 5 we get

$$2(n-s-1)^2,\ \text{for } n-s\geqslant 4,$$

(vi) $\alpha\cap\{1,\cdots,s\}=\{1,j\}$, $1<j<s$, $|\beta\cap\{1,\cdots,s\}|=1$ or
$|\alpha\cap\{1,\cdots,s\}|=1$, $\beta\cap\{1,\cdots,s\}=\{i,s\}$, $1\leqslant i\leqslant s$:
Since every row and column of A' has at least two $1'$s, we get at least

$$2(s-1)\times(2)\times 2(n-s-1)=8(s-1)(n-s-1).$$

(vii) $1\in\alpha\subseteq\{1,\cdots,s\}$, $|\beta\cap\{1,\cdots,s\}|=2$ or

$s \in \beta \in \{1, \cdots, s\}$, $|\alpha \cap \{1, \cdots, s\}| = 2$: By Lemma 4 we get at least

$$2\binom{2s-1-2}{2} = (2s-3)(2s-4).$$

Adding the estimates we obtain the following cubic polynomial in n:

$$\sigma(C_3^*(A)) \geqslant \frac{4n^3 - 18n^2 + (12s+11)n + (-6s^2 - 30s + 30)}{3}.$$

The difference between the estimate for $\sigma_3(C_3^*(A))$ above and that given in (9) is

$$\frac{(12s-9)n - (-6s^2 - 30s + 30)}{3}. \tag{10}$$

Since $s \leqslant n-3$, $n \geqslant s+3$ and

$$(12s-9)n \geqslant (12s-9)(s+3) = 12s^2 + 27s - 27.$$

Using this estimate in (10), we see that (10) is positive, since $s \geqslant 1$. It follows that $\sigma(C_3^*(A)) > 2n(n-2)(2n-5)/3$, completing the induction and proving the theorem.

In our proof of Theorem 4, Lemma 5 was crucial. It would seem that to prove that $\sigma(C_r^*(A)) \geqslant \sigma(C_r^*(F_n))$ for $r > 3$, one would need some analogue of Lemma 5. In particular to prove $\sigma(C_4^*(A)) \geqslant \sigma(C_4^*(F_n))$, one would like to be able to estimate $\sigma(C_3^*(B))$ where B is obtained from a nearly decomposable matrix by crossing out a row or column. But a good estimate seems to be difficult to obtain, since one cannot simply count the number of sets of three independent 1′s.

References

[1] R. A. Brualdi, Permanent of the product of doubly stochastic matrices, Proc. Camb. Phil. Soc., 62(1966), 643–658.

[2] R. A. Brualdi, Matrices permutation equivalent to irreducible matrices and applications, Lin Multilin. Alg., 7(1979), 1–12.

[3] R. A. Brualdi and P. M. Gibson, Convex polyhedra of doubly stochastic matrices I. Applications of the permanent function, J. Combin. Theory 22(1977), 194–230.

[4] D. J. Hartfiel, A simplified form for nearly reducible and nearly decomposable matrices, Proc. Amer. Math. Soc. 24(1970), 388–393.

[5] D. J. Hartfiel, On constructing nearly decomposable matrices, Proc. Amer. Math. Soc. 27 (1971), 222–228.

[6] C. C. MacDuffee, The theory of matrices, Chelsea, New York.

[7] H. Minc, Permanents, Addison-Wesley, Reading, 1978.

[8] H. J. Ryser, Inepualities of compound and induced matrices with applications to combinatorial analysis, Ill. J. Math. 2(1958), 240–253.

[9] H. J. Ryser, Combinatorial Mathematics, Math. Assoc. of America, Washington, D. C., 1963

[10] R. S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, 1962.