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$\underline{\theta}$-refinability and some related properties*

Wu Lisheng

(Suzhou University)

Definition 1. A space is θ-refinable, if every directed open cover of the space has a pointwise star-refining sequence by open covers.

It is well-known, that both strict p and θ-refinability imply $\underline{\theta}$-refinability, but we don't know if strict p (or $\underline{\theta}$-refinability) implies θ-refinability[1], and our paper is just discussing some facts related to this problem.

Theorem 1. Let X be a completely regular locally compact space, then the following conditions are mutually equivalent:

(i) X is strict p,

(ii) The collection $\mathscr{K}$ which consists by all compact subsets of X has a pointwise star-refining sequence by open covers of X,

(iii) X is θ-refinable,

(iv) X is $\underline{\theta}$-refinable.

Proof. (i)⇒(ii)⇒(iii) are due to [2], (iii)⇒(iv) is evident,

(iv)⇒(ii) Let X be a locally compact $\underline{\theta}$-refinable space. For each $x\in X$, there exists an open neighborhood O_x such that $\overline{O_x}$ is compact. Let $\mathscr{O}=\{O_x: x\in X\}$, then $\mathscr{O}^F$ is a directed open cover of X, so it has a pointwise star-refining sequence $\{\mathscr{U}_n: n\in N\}$ by open covers of X. For each $x\in X$, there exists $n\in N$ and $O\in\mathscr{O}^F$ such that $st(x,\mathscr{U}_n)\subset O\subset\overline{O}\in\mathscr{K}$, so $\{\mathscr{U}_n: n\in N\}$ is also a pointwise star-refining sequence of $\mathscr{K}$.

(iii)⇒(ii)⇒(i) Let $\{\mathscr{U}_n: n\in N\}$ be a pointwise star-refining sequence of $\mathscr{K}$. where each $\mathscr{U}_n$ is an open cover of X. No loss of generality, we may assume $\mathscr{U}_{n+1}$ refines $\mathscr{U}_n$.

Suppose $x_n\in st(x,\mathscr{U}_n)$. then there exist $n\in N$ and $K\in\mathscr{K}$ such that $\{x_m\}_{m\geq n}\subset st(x,\mathscr{U}_n)\subset K$, so $\{x_n\}_{n\in N}$ has a cluster point.this shows X is a ω_Δ space. Now X is strict p in view of [3, theorem 1.7].

Lemma 1. Every $\underline{\theta}$-refinable space is θ-expandable.

Proof. Let $\mathscr{F}=\{F\}$ be a locally finite closed colleetion of X, Define $\mathscr{G}=\{X-(\mathscr{F}-\tilde{\mathscr{F}})^*: \tilde{\mathscr{F}}\subset\mathscr{F}, |\tilde{\mathscr{F}}|<\infty\}$, then $\mathscr{G}$ is a directed open cover of X, so it has a pointwise star-refining sequence $\{\mathscr{U}_n: n\in N\}$ by open covers of X.

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For each $F \in \mathscr{G}$, Let $\mathscr{O}_n(F) = st(F, \mathscr{U}_n)$, then for each $n \in N$, $\mathscr{O}_n = \{O_n(F): F \in \mathscr{G}\}$ is a open expansion of $\mathscr{G}$.

For each $x \in X$, there exists $n \in N$ and $\tilde{\mathscr{G}} \subset \mathscr{G}$, $|\tilde{\mathscr{G}}| < \infty$, such that $st(x, \mathscr{U}_n) \subset X(\mathscr{G} - \tilde{\mathscr{G}})^*$. If $F \notin \tilde{\mathscr{G}}$, then $st(x, \mathscr{U}_n) \cap F = \emptyset$, and thus $x \notin st(F, \mathscr{U}_n) = \mathscr{O}_n(F)$. so $|(\mathscr{O}_n)_x| \leq |\tilde{\mathscr{G}}| < \infty$. i.e. X is θ-expandable.

Corollary 1. Every $\underline{\theta}$-refinable space is countably metacompact.

Proof. Because countably metacompactness is equal to $\aleph_0$-θ-expandability.

Theorem 2. Every $\underline{\theta}$-refinable space is ultrapure.

Proof. Suppose $\mathscr{U}$ is an open cover of $\underline{\theta}$-refinable space X. $\mathscr{U}^F$ has a pointwise star-refining sequence $\{\mathscr{U}_n : n \in N\}$ by open covers of X. Let $D_n = \{x \in X: st(x, \mathscr{V}_n) \subset W$ for some $W \in \mathscr{U}^F\}$. then $\cup\{D_n : n \in N\} = X$.

Let $\mathscr{D}_n = \{D_n \cap v : v \in \mathscr{D}_n\}$, then (i) $\cup\{\mathscr{D}_n^* : n \in N\} = \cup\{D_n : n \in N\} = X$, (ii) each member of $\mathscr{D}_n$ is an open subset of subspace $\mathscr{D}_n^*$, (iii) $\{st(x, \mathscr{D}_n) : n \in N, x \in \mathscr{D}_n^*\}$ refines $\mathscr{U}^F$, this shows X is ultrapure,

Corollary 2. Every $\underline{\theta}$-refinable space is isocompact.

Theorem 3. Every σ-orthcompact. $\underline{\theta}$-refinable space is θ-refinable.

Proof. Let X be a σ-orthcompact. $\underline{\theta}$-refinable space, $\mathscr{U}$ be a directed open cover of X. In view of corollary 1 and [4, proposition 3.1], X is orthcompact. $\mathscr{U}$ has a pointwise star-refining sequence $\{\mathscr{V}_n : n \in N\}$ by open covers of X. Each $\mathscr{V}_n$ has an interior-preserving refinement $\mathscr{W}_n$. Now $\mathscr{U}$ has a pointwise star-refining sequence $\{\mathscr{W}_n : n \in N\}$ by interior-preserving open covers of X, so X is θ-refinable in view of [5].

Corollary 3. Every σ-orthocompact strict p space is θ-refinable.

Theorem 4. Every pointwise-star-orthocompact $\underline{\theta}$-refinable space is weak orthocompact.

Proof. Let X be a pointwise star-orthocompact θ-refinable space, $\mathscr{U}$ be a directed open cover of X. Then $\mathscr{U}$ has a pointwise star-refining sequence $\{\mathscr{V}_n : n \in N\}$ by open covers of X. For each $n \in N$, there exists an interior-preserving open cover $\mathscr{W}_n = \{W(n, x) : x \in X\}$ such that for each $x \in X$, $x \in W(n, x) \subset st(x, \mathscr{V}_n)$.

For each $x \in X$, there exists $n_x \in N$ and $U(x) \in \mathscr{U}$ such that $st(x, \mathscr{V}_{n_x}) \subset U(x)$.

Let $X_n = \{x \in X : n_x = n\}$, $\tilde{\mathscr{W}}_n = \{W(n, x) : x \in X_n\}$. then $\cup\{\tilde{\mathscr{W}}_n : n \in N\}$ is a σ-interior-preserving open refinement of $\mathscr{U}$. Similar to the proof of [4, propersition 3.1], We can show $\mathscr{U}$ has a interior-preserving open refinement, i.e., X is weak orthocompact.

Theorem 5. Every pointwise star-orthocompact strict p space is $\underline{\theta}$-refinable.

Proof. Let X be a pointwise star-orthocompact strict p space, $\mathscr{U}$ be a open cover of X. There exists a sequence $\{g_n : n \in N\}$ by open covers of X such that:

(i) For each $x\in X$, $P_x=\bigcap\{st(x,\mathcal{G}_n):n\in N\}$ is a compact set,

(ii) The family $\{st(x,\mathcal{G}_n):n\in N\}$ is a neighborhood base for the set P_x.

(iii) g_{n+1} refines g_n

For each $n\in N$, there exists an interior-preserving open cover $\mathcal{O}_n=\{O_{n,x}:x\in X\}$ Such that $x\in O_{n,x}\subset st(x,g_n)$. Let $h(n,x)=\bigcap(\mathcal{O}_n)_x$, then $\{h(n,x):x\in X\}$ is also an interior-preserving collection.

Define $\mathcal{F}_n=\{X-h(n,x):x\in X\}\cup\{X\}$, $\mathcal{F}=\bigcup\{\mathcal{F}_n:n\in N\}$, each $\mathcal{F}_n$ is a closure-preserving closed cover of X.

For each $x\in X$, let $c(x)=\bigcap(\mathcal{F})_x=\bigcap\{\bigcap(\mathcal{F}_n)_x:n\in N\}$. We claim for each $n\in N$, $\bigcap(\mathcal{F}_n)_x\subset st(x,g_n)$. In fact, for each $z\notin st(x,g_n)$, we have $x\notin h(n,z)$, otherwise $x\in h(n,z)\subset O_{n,z}\subset st(z,\mathcal{G}_n)$ this would implies $z\in st(x,g_n)$, a contradiction.

So we have $z\in h(n,z)\subset\bigcup\{h(n,y):x\notin h(n,y)\}$, i.e.

$$z\notin X-\bigcup\{h(n,y):x\notin h(n,y)\}=\bigcap(\mathcal{F}_n)_x.$$

Thus $c(x)\subset p_x$ for each $x\in X$, and $c(x)$ is also a compact set.

For each $x\in X$, there exists $U_x\in\mathcal{U}$ such that $c(x)\subset P_x\subset U_x$, then there exists $n_x\in N$ such that $c(x)\subset p_x\subset st(x,g_{n_x})\subset U_x$, so $c(x)\subset\bigcap(\mathcal{F}_n)_x\subset st(x,g_n)\subset U_x$.

Let $X_n=\{x\in X:n_x=n\}$, $\text{Ж}_n^x=\{\bigcap(\mathcal{F}_n)_x:x\in X_n\}$. For each $n\in N$, Ж_n is a partial refinement of $\mathcal{U}$, and Ж_n is a closed-preserving closed collection. In fact, if $y\in\overline{\bigcup\{\bigcap(\mathcal{F}_n)_x:x\in\tilde{X}\subset X\}}$, then $h(n,y)$ is an open neighborhood of y, and there exists $x\in\tilde{X}$ such that $h(n,y)\cap(\bigcap(\mathcal{F}_n)_x)\neq\emptyset$. i.e. $h(n,y)\cap(X-$ $-\bigcup\{h(n,z):x\notin h(n,z)\}\neq\emptyset$, so $x\in h(n,y)$.

For each $z\in X$, if $x\notin h(n,z)$, then $y\notin h(n,z)$, otherwise from $y\in h(n,z)$ would implies $x\in h(n,y)\subset h(n,z)$, this is impossible. Thus $y\in\bigcap(\mathcal{F}_n)_x$, i.e. Ж_n is a closure-preserving collection. Now $\mathcal{U}$ has a σ-cloure-preserving closed refinement $\text{Ж}=\bigcup\{\text{Ж}_n:n\in N\}$, i.e. X is θ-refinable.

Corollary 4. Every pointwise star-orthocompact strict p space is also an orthocompact $\Sigma^{\#}$space.

Definition 2. A open cover $\mathcal{V}=\bigcup\{\mathcal{V}_n:n\in N\}$ is called a boundly weak $\overline{\theta}$-cover. if

(i) $\{\mathcal{V}_n^*:n\in N\}$ is point-finite.

(ii) There exists $k\in N$ such that for each $x\in X$, there exists $n\in N$ such that $ord(x,\mathcal{V}_n)\leqslant k$.

Theorem 6. Let X be a space, then the following conditions are mutually equivalent.

(i) X is discrete θ-expandable, (ii) Every boundly weak $\overline{\theta}$-cover of X hasa θ-sequence by open refinement. (iii) Every c-cover has a θ-sequence by open refinement.

Proof. (i)$\Rightarrow$(ii). Suppose X is discrete θ-expandable, $\mathcal{U}$ is a boundly weak $\overline{\theta}$-cover of X. Then $\mathscr{U}$ has a refinement $\mathscr{P}=\bigcup\{\mathscr{P}_n: n\in N\}$ such that for each $n\in N$, $\{p-\bigcup\{\mathscr{P}^*_i: 1\leqslant i\leqslant n\}: P\in\mathscr{P}_n\}$ is a discrete closed collection in $X-\bigcup\{\mathscr{P}^*_i: 1\leqslant i<n\}$ (See [7])

For each $P\in\mathscr{P}$, choose an $U_P\in\mathscr{U}$ such that $P\subset U_P$. Since $\mathscr{P}_1$ is a discrete closed collection in X, $\mathscr{P}_1$ has open expansion $\mathscr{V}_n=\{V_{n_1,P}: P\in\mathscr{P}_1\}$ for each $n_1\in N$ such that For each $P\in\mathscr{P}_1$, $P\subset V_{n,P}\subset U_P$, and for each $x\in X$, there exists n_1 such that $\operatorname{ord}(x,\mathscr{V}_{n_1})<\infty$.

Suppose for each $1\leqslant k<j$, each $(n_1,\cdots,n_k)\in N^k$, we have constructed open collection $\mathscr{V}_{n_1\cdots,n_k}=\{V_{n_1,\cdots,n_k,P}: P\in\mathscr{P}_k\}$ such that $P-\bigcup\{\mathscr{V}^*_{n_1,\cdots,n_i}: 1\leqslant i<k\}\subset V_{n_1,\cdots,n_k,P}\subset U_P-\bigcup\{\mathscr{P}^*_i: 1\leqslant i<k\}$. For each $P\in\mathscr{P}_k$, and for each x, each $(n_1,\cdots,n_{k-1})\in N^{k-1}$, there exists n_k such that $\operatorname{ord}(x, \ _{n_1,\cdots,n_k})<\infty$. We now construct the collection $\mathscr{V}_{n_1,\cdots,n_{k+1}}$.

Firstly, in view of the inductive conditions, for each $1\leqslant k\leqslant j$, holds $\bigcup\{\mathscr{P}^*_i: 1\leqslant i\leqslant k\}\subset\bigcup\{\mathscr{V}^*_{n_1,\cdots,n_i}: 1\leqslant i\leqslant k\}$. In fact, it holds for $k=1$ is evident, suppose $\bigcup\{\mathscr{P}^*_i: 1\leqslant i\leqslant s\}\subset\bigcup\{\mathscr{V}^*_{n_1,\cdots,n_i} 1\leqslant i\leqslant s\}$ for each $s<s+1<j$, then $\bigcup\{\mathscr{V}^*_{n_1,\cdots,n_i} 1\leqslant i\leqslant s+1\}=\bigcup\{\mathscr{V}^*_{n_1,\cdots,n_i}: 1\leqslant i\leqslant s\}\bigcup\mathscr{V}^*_{n_1,\cdots,n_{s+1}}\supset\bigcup\{\mathscr{V}^*_{n_1,\cdots,n_i}: 1\leqslant i\leqslant s\}\bigcup(\mathscr{P}^*_{s+1}-\bigcup\{\mathscr{V}^*_{n_1,\cdots,n_i}: 1\leqslant i\leqslant s\})=\bigcup\{\mathscr{V}^*_{n_1,\cdots,n_i}: 1\leqslant i\leqslant s\}\bigcup\mathscr{P}^*_{s+1}\supset\bigcup\{\mathscr{P}^*_i: 1\leqslant i\leqslant s+1\}$. Thus we have $\bigcup\{\mathscr{P}^*_i: 1\leqslant i<j\}\subset\bigcup\{\mathscr{V}^*_{n_1,\cdots,n_i}: 1\leqslant i<j\}$.

$\mathscr{P}^1_j=\{P-\bigcup\{\mathscr{V}^*_{n_1,\cdots,n_i}: 1\leqslant i<j-1\}: P\in\mathscr{P}_j\}$ is a discrete closed collection in X, so there exists open collections $\mathscr{V}_{n_1,\cdots,n_j}=\{V_{n_1,\cdots,n_j,P}: P\in\mathscr{P}_j\}$ such that $P-\bigcup\{\mathscr{V}^*_{n_1,\cdots,n_i}: 1\leqslant i<j-1\}\subset V_{n_1,\cdots,n_j,P}\subset U_P-\bigcup\{\mathscr{P}^*_i: 1\leqslant i<j\}$ for each $P\in\mathscr{P}_j$, and for each $x\in X$ each $(n_1,\cdots,n_{j-1})\in N^{j-1}$, there exists n_j such that $\operatorname{ord}(x, \mathscr{V}_{n_1,\cdots,n_j})<\infty$.

Thus we can construct for each $k\in N$, each $(n_1,\cdots,n_k)\in N^k$ an open collection $\mathscr{V}_{n_1,\cdots,n_k}$ such that

(i) $\mathscr{V}_{n_1,\cdots,n_k}$ is a partial refinement of $\mathscr{U}$,

(ii) $\mathscr{V}_{n_1,\cdots,n_k}\bigcap(\bigcup\{\mathscr{P}^*_i: 1\leqslant i<k-1\})=\emptyset$,

(iii) $\bigcup\{\mathscr{P}^*_i: 1\leqslant i\leqslant k\}\supset\bigcup\{\mathscr{V}^*_{n_1,\cdots,n_i}: 1\leqslant i\leqslant k\}$.

Define $\mathscr{V}_{(n_1,\cdots,n_m)}=\bigcup\{\mathscr{V}_{n_1,\cdots,n_i}: i\in N,\ n_i=n_m \text{ when } i\geqslant m\}$

$$\mathscr{V}=\{\mathscr{V}_{(n_1,\cdots,n_m)}: m\in N,\ \ (n_1,\cdots,n_m)\in N^m\}.$$

then $\mathscr{V}$ is a θ-sequence. In fact, for each $x\in X$, there exists $m\in N$ such that $x\in\mathscr{P}^*_m$. there exists n_1 such that $\operatorname{ord}(x,\mathscr{V}_{n_1})<\infty$, then there exists $n_2,\cdots,n_m$ one after one such that for each $1\leqslant i\leqslant m$ $\operatorname{ord}(x,\mathscr{V}_{n_1,\cdots,n_i})<\infty$, so $\operatorname{ord}(x,$

$$\mathscr{V}_{(n_1,\cdots,n_m)})=\sum_{i=1}^{m}\operatorname{ord}(x,\mathscr{V}_{n_1,\cdots,n_i})<\infty.$$

i.e. $\mathscr{U}$ has a θ-sequence by open refinement.

(ii)$\Rightarrow$(iii). Suppose $\mathscr{U}=\{U_a: a\in A\}$ is a C-cover, *i.e.*, for each $a\in A$, $U_a=\bigcap_{\beta\neq a}\bigcup_{\gamma\neq\beta}U_\gamma$. Define $F_a=X-\bigcup_{\beta\neq a}U_\beta$, then for each $a\in A$, $F_a\subset U_a$ and X-

$\bigcup\{F_a: a\in A\}\subset\bigcap\{U_a: a\in A\}$. Let $\mathscr{V}_1=\{X-\bigcup\{F_a: a\in A\}\}$, $\mathscr{V}_2=\mathscr{U}$, then $\mathscr{V}=\mathscr{V}_1\cup\mathscr{V}_2$ is a boundly weak-$\bar{\theta}$-cover which refines $\mathscr{U}$. so $\mathscr{U}$ has a θ-sequence by open refinement.

(iii) $\to$ (i) is due to [8, Lemma 2, 7].

Corollary 5. Let X be a space which has property B(D, ω), then

(1) X is collectionwise normal iff X is paracompact,

(2) X is collectionwise subnormal iff X is subparacompact,

(3) X is almost discrete expandable iff X is metacompact

Similar to Theorem 6, we have

Thcorem 7 Let X be a θ-expandable space, $\mathscr{U}$ be a open cover of X and has a refinement $\mathscr{P}=\bigcup\{\mathscr{P}_n: n\in N\}$ such that, for each $n\in N$, $\{P-\bigcup\{\mathscr{P}_i^*: 1\leqslant i< n\}: P\in\mathscr{P}_n\}$ is a locally finite closed collection in $X-\bigcup\{\mathscr{P}_i^*: 1\leqslant i<n\}$, then $\mathscr{U}$ has a θ-sequence by open refinement.

Remark Theorem 7 shows that if $\underline{\theta}$-refinability implies property B(LF, ω), then $\underline{\theta}$-refinability is equal to θ-refinability.

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