紧Lie群上Fourier级数的Gauss-Weierstrass型平均的逼近性质*

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设G为连通、半单的紧Lie群．$f\in L^1(G)$的Fourier级数

$$f\sim\sum_{\lambda\in\Lambda(G)}d_\lambda\chi_\lambda * f(g)$$

的Gauss-Weierstrass型平均为

$$G_R^a(f,g)=\sum_{\lambda\in\Lambda(G)}e^{-|\lambda+\beta|^a R^{-a}}d_\lambda\chi_\lambda * f(g),\quad a>0$$

当G为可交换群时，$G_R^a(f,g)$实际上就是多重Fourier级数的Gauss-Weierstrass型平均，Б.И.Голубов 在文[1]中研究了它的一些逼近性质．本文在G为不可交换群时对$G_R^a(f,g)$作了研究，推广了[1]中的主要结果，并给出了$G_R^a(f,g)$对可微函数的一致逼近阶．

一．定义与结果

设L为G的Lie代数，T为G的极大环群，H为T的Lie代数$\dim G=n$，$\dim T=l$，$a_1,\cdots,a_m$为H上全体正根，则$n=2m+l$．设$B(\cdot,\cdot)$为L上由Killing形式构成的不变内积，δ为G上由$B(\cdot,\cdot)$产生的不变Riemann度量，$\rho(e,\exp h)=[B(h,h)]^{1/2}\equiv|h|$，$h\in Q$．这里$e$为G的幺元，exp为L到G的指数映射，Q为T的中心在0点的基本区域．

设$f\in L^p(G)$，定义f的连续模及光滑模如下：

$$\omega(f,t;p)=\sup\{\|R_gf-f\|_p;\ \rho(g,e)\le t\},$$

$$\omega_2(f,t;p)=\sup\{\|R_{g^2}f-2R_gf+f\|_p;\ \rho(g,e)\le t\},$$

这里$R_gf(g_1)=f(g_1g)$，$1\le p\le\infty$，当$p=\infty$时就简记为$\omega(f,t)$和$\omega_2(f,t)$．

设$X_1,\cdots,X_n$为L关于$B(\cdot,\cdot)$的一组标准正交基．若f为k次可微，且对任何$X^{(1)},\cdots,X^{(k)}\in L$有$X^{(k)}X^{(k-1)}\cdots X^{(1)}f\in L^p(G)$，则定义

$$\omega(f^{(0)},t;p)=\omega(f,t;p),\quad \omega(f^{(k)},t;p)=\sum_{i_1,\cdots,i_k=1}^{n}\omega(X_{i_1}\cdots X_{i_k}f,t;p)$$

当$p=\infty$时简记为$\omega(f^{(k)},t)$．这里$Xf(g)=\{\frac{d}{dt}f(g\exp tX)\}_{t=0}$，$g\in G$，$X\in L$．

设ω为凸连续模，r为非负整数，$0<\gamma\le 2$，$0<\sigma\le 1$，$1\le p\le\infty$，定义函数类如下：

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$H_\omega(G)=\{f\in C(G);\ \omega(f,t)\leqslant\omega(t)\}$，$\mathrm{Lip}_1\sigma=\{f\in C(G);\ \omega(f,t)\leqslant t^\sigma\}$，

$H_p^{2r+\gamma}(G)=\{f\in L^p(G)\ |X_{i_1}\cdots X_{i_{2r-1}}f$ 绝对连续，$X_{i_1}\cdots X_{i_{2r}}f\in H_p^\gamma(G)$，$i_1,\cdots,i_{2r}=1,2,\cdots,n\}$，$H_p^\gamma(G)=\{f\in L^p(G);\ \omega_2(f,t;p)=O(t^\gamma),\ t\to 0\}$.

以下恒用C表示不同的绝对常数．我们的主要结果如下：

定理 1 设$f\in H_p^{2r+\gamma}(G)$，则当$R\to\infty$时有

$$\|G_R^a(f)-f\|_p=\begin{cases}O(R^{-2r-\gamma}), & a>2r+\gamma,\\ O(R^{-2r-\gamma}\log R), & a=2r+\gamma<2r+2,\\ O(R^{-2r-2}), & a=2r+\gamma=2r+2,\\ O(R^{-a}), & a<2r+\gamma.\end{cases}$$

定理 2 设$f\in C^k(G)$，$a>k$，则

$$\|G_R^a(f)-f\|_\infty\leqslant\begin{cases}CR^{1-a}\omega(f^{(k)},1/R), & k<a<k+1,\\ CR^{-k}\omega(f^{(k)},1/R)\log R, & a=k+1\ \text{为奇数},\\ CR^{-k}\omega(f^{(k)},1/R), & a>k+1\ \text{或}\ a=k+1\ \text{为偶数}.\end{cases}$$

定理 3 $G_R^a(f)$逼近f的饱和阶为R^{-a}.

定理 4 设$\varphi_{n,a}(|x|)$为$e^{-|x|^a}$的Fourier变换，$a(n,l)=a_0\{(2\pi)^{\frac{l}{2}}|W||\prod_{j=1}^m B(\beta,a_j)|\}^{-1}$，$b(n,a)=\pi^{-n/2}2^{a-1}a\Gamma(\frac{n+a}{2})\cdot[\Gamma(1-\frac{a}{2})]^{-1}$，这里$a_0=\int_{|\eta|=1}[\prod_{j=1}^m B(\eta,a_j)]^2\mathrm{d}\sigma(\eta)$，$\beta=\frac{1}{2}\sum_{j=1}^m a_j$，$|W|$为G关于T的Weyl 群之阶，则

(i)
$$\sup_{f\in\mathrm{Lip}_1\sigma}\|G_R^a(f)-f\|_\infty=2\sup_{f\in H_\infty^\sigma(G)}\|G_R^a(f)-f\|_\infty$$
$$=\begin{cases}a(n,l)b(n,a)R^{-\sigma}\log R+O(R^{-\sigma}), & a=\sigma,\\ \{a(n,l)\int_0^\infty\varphi_{n,a}(t)t^{n+\sigma-1}\mathrm{d}t\}R^{-\sigma}+O(R^{-a}), & \sigma<a\leqslant 1,\\ \{a(n,l)\int_0^\infty\varphi_{n,a}(t)t^{n+\sigma-1}\mathrm{d}t\}R^{-\sigma}+O(R^{-\sigma-a+1}), & 1<a\leqslant 2,\end{cases}$$

(ii)
$$\sup_{f\in H_\omega(G)}\|G_R^a(f)-f\|_\infty=a(n,l)\int_0^R\omega(\frac{t}{R})\varphi_{n,a}(t)t^{n-1}\mathrm{d}t+O(R^{1-a}\omega(R^{-1})),\ 1<a\leqslant 2.$$

二．几个引理

引理 1 若a不为偶数，则当$t\to\infty$时有$\varphi_{n,a}(t)=-\pi^{-\frac{n}{2}}\Gamma(\frac{a+n}{2})[\Gamma(-\frac{a}{2})]^{-1}2^a t^{-a-n})+O(t^{-a-n})$．若$a$为偶数，则对任何$q>0$，有$\varphi_{n,a}(t)=O(t^{-q})$，$t\to\infty$.

证明见[1].

引理 2 设f为k次可微．令$\varphi_{g,g_1,\eta}(t)=f(gg_1(\exp t\eta)g_1^{-1})$，$g,g_1\in G$，$\eta\in H$，$|\eta|=1$，$t\geqslant 0$，则 $\varphi_{g,g_1,\eta}^{(k)}(t)=\sum_{j_1,\cdots,j_k=1}^n a_{j_1}(g_1,\eta)\cdots a_{j_k}(g_1,\eta)\cdot X_{j_1}\cdots X_{j_k}f(gg_1(\exp t\eta)g_1^{-1})$.

其中$a_j(g_1,\eta)$ $(j=1,2,\cdots,n)$ 满足 $\sum_{j=1}^n|a_j(g_1,\eta)|^2=1$.

证明 设Ad 为G的伴随表示，则对任何$g\in G$，Adg是L的自由构．设

$$\mathrm{Ad}g(X)=\sum_{j=1}^n a_j(g,X)X_j,\ g\in G,\ X\in L, \tag{1}$$

则可得 $$g_1(\exp t\eta)g_1^{-1}=\exp[t\sum_{j=1}^{n}a_j(g_1,\eta)X_j].\tag{2}$$

由于 $$\frac{d}{dt}f(g\exp tX)=Xf(g\exp tX),\ g\in G,\ X\in L,\tag{3}$$

故由(2)可得 $\varphi'_{g,g_1,\eta}(t)=\sum_{j=1}^{n}a_j(g_1,\eta)X_jf(gg_1(\exp t\eta)g_1^{-1})$.

由Killing形式的性质及(1)可得

$$\sum_{j=1}^{n}|a_j(g_1,\eta)|^2=|\eta|^2=1.\tag{4}$$

运用归纳法即得引理2.

引理3 设r、s为非负整数，$2s+2\leqslant 2r$．若$f\in H_p^{2r+\gamma}(G)$，则$f\in H_p^{2s+2}(G)$.

证明 由Cartan的极大环群定理及指数映射的性质，对$g\in G$，存在唯一的$h\in Q$，及$g'\in G$，使$g=g'(\exp h)g'^{-1}$．由此及(2)—(4)，利用Lagrange中值公式及Haar测度的不变性可得 $\|R_{g_1^2}(X_{i_1}\cdots X_{i_{2l}}f)-2R_{g_1}(X_{i_1}\cdots X_{i_{2l}}f)+X_{i_1}\cdots X_{i_{2l}}f\|_p\leqslant 2\sum_{i,j=1}^{n}\|X_iX_jX_{i_1}\cdots X_{i_{2l}}f\|_p\cdot\delta(g_1,e)^2$. 引理3得证.

引理4 记$H_1^{(k)}(t)=-\int_0^{\infty}\varphi_{n,a}(\tau)\tau^{n+k-1}d\tau$，$H_j^{(k)}(t)=-\int_t^{\infty}H_{j-1}^{(k)}(\tau)d\tau$，$j=2,3,\cdots,k$ $k=0,1,2,\cdots$，则

(i) $|H_j^{(k)}(t)|\leqslant C$，$a>j+k-1$，$j=1,2,\cdots$，$k=0,1,2,\cdots$；

(ii) $|H_j^{(k)}(t)|\leqslant Ct^{-a+j+k-1}$，$t>1$，$a>j+k-1$；

(iii) 若$a>2r$，r为自然数，则 $H_3^{(0)}(0)=H_5^{(0)}(0)=\cdots=H_{2r+1}^{(0)}(0)=0$, $H_2^{(k)}(0)=H_4^{(k)}(0)=\cdots=H_{2r-k+1}^{(k)}(0)=0$，$k$为奇数，$H_1^{(k)}(0)=H_3^{(k)}(0)=\cdots=H_{2r-k+1}^{(k)}(0)=0$，$k$为偶数.

证明 由[1]，$\varphi_{n,a}(t)=(2\pi)^{-\frac{n}{2}}t^{1-\frac{n}{2}}\int_0^{\infty}e^{-\tau^a}\tau^{\frac{n}{2}}J_{n/2-1}(t\tau)d\tau$， 故有

$$|\varphi_{n,a}(t)|\leqslant C.\tag{5}$$

由引理1对j用归纳法可得(ii)．由(5)、(ii)及引理1，对j用归纳法可得(i)．由[1]，$G_R^a(\varphi,x)=\int_0^{\infty}\varphi_x(\frac{t}{R})dH_1^{(0)}(t)$，$x\in T^n$．$\varphi\in L^1(T^n)$，$T^n$为$n$维环群．取$\varphi_{(x)}=e^{ix_1}$，则由[2]，$\varphi_x(t)=e^{ix}V_{\frac{n}{2}-1}(t)$，这里$V_s(t)=t^{-s}J_s(t)$，$J_s(t)$为$s$阶Bessel函数．于是可得

$$V_{\frac{n}{2}-1}(0)H_1^{(0)}(0)-\frac{1}{R}\int_0^{\infty}V'_{\frac{n}{2}-1}(\frac{t}{R})H_1^{(0)}(t)dt=e^{-R^{-a}}.$$

由于$H_1^{(0)}(t)\in L^1(0,\infty)$，故令$R\to\infty$可得$V_{\frac{n}{2}-1}(0)H_1^{(0)}(0)=1$．于是分部积分可得 $\int_0^{\infty}V''_{\frac{n}{2}-1}(\frac{t}{R})H_2^{(0)}(t)dt=O(R^{-a+2})$．注意$V''_{n/2-1}(0)\neq 0$，所以当$a>2$时令$R\to\infty$得 $H_3^{(0)}(0)=0$．由Bessel函数的级数表达式易知 $V_{\frac{n}{2}-1}^{(2k-1)}(0)=0$，$V_{\frac{n}{2}-1}^{(2k)}(0)\neq 0$，由此与一样可得$a>2r$时有

$$H_3^{(0)}(0)=H_5^{(0)}(0)=\cdots=H_{2r+1}^{(0)}(0)=0. \tag{6}$$

用归纳法可证

$$H_j^{(k)}(t)=tH_j^{(k-1)}(t)-jH_{j+1}^{(k-1)}(t),\quad j,k=1,2,\cdots. \tag{7}$$

由(6)、(7)用归纳法可得(iii). 引理4得证.

三. 定理的证明

由于篇幅的限制，我们仅给出定理1,2的证明.

定理1的证明 与[3]中类似可得

$$G_R^a(f,g)-f(g)=C_0R^n\int_H[\psi_g(h)-f(g)]\Delta(h)D(h)\varphi_{n,a}(R|h|)\,dh,$$

其中$C_0=(-1)^m[(2\pi)^{l/2}|W|D(\beta)]^{-1}$, $\psi_g(h)=\int_{G/T}f(gg_1(\exp h)g_1^{-1})\,d\overline{g_1}$, $\Delta(h)=\prod_{j=1}^m 2i\sin\frac{1}{2}B(h,a_j)$, $D(h)=\prod_{j=1}^m B(h,a_j)$. 易见$\Delta(-h)D(-h)=\Delta(h)D(h)$, 故有

$$G_R^a(f,g)-f(g)=\frac{C_0}{2}R^n\int_H[\psi_g(h)+\psi_g(-h)-2f(g)]\Delta(h)D(h)\psi_{n,a}(R|h|)\,dh$$

$$=\frac{C_0}{2}R^n\left\{\int_0^1+\int_1^\infty\right\}\left\{\int_{|\eta|=1}[\psi_g(t\eta)+\psi_g(-t\eta)-2f(g)]\Delta(t\eta)D(\eta)\,d\sigma(\eta)\right\}$$

$$\cdot t^{m+l-1}\varphi_{n,a}(Rt)\,dt\equiv I_R^{(1)}(f)+I_R^{(2)}(f). \tag{8}$$

由Fubini定理及Holder不等式可得

$$\left\{\int_G|\psi_g(h)+\psi_g(-h)-2f(g)|^p dg\right\}^{1/p}\leqslant\begin{cases}\omega_2(f,\ \sup\limits_{g\in G}\rho(g,e);p),& h\in H,\\ \omega_2(f,\ |h|;p),& h\in Q.\end{cases} \tag{9}$$

由此利用引理1可得

$$\|I_R^{(2)}(f)\|_p\leqslant CR^{-a}. \tag{10}$$

由幂级数展开可得

$$\Delta(t\eta)=(2i)^m\sum_{s=0}^\infty(-1)^s\sum_{\substack{j_1+\cdots+j_m=m+2s\\ j_1,\cdots,j_m\in\{1,3,5,\cdots\}}}\prod_{k=1}^m\frac{1}{j_k!}B(\eta,a_k)^{j_k}\left(\frac{t}{2}\right)^{m+2s}$$

对一切$t\in[0,\infty)$绝对收敛(关于η一致). 由此可得

$$I_R^{(1)}(f)=\sum_{j=0}^{r-1}\frac{C_0}{2}R^n\int_0^1\left\{\int_{|\eta|=1}[\psi_g(t\eta)+\psi_g(-t\eta)-2f(g)]C_j(\eta)D(\eta)\,d\sigma(\eta)\right\}\cdot$$

$$\cdot t^{n+2j-1}\varphi_{n,a}(Rt)\,dt+\frac{C_0}{2}R^n\int_0^1\left\{\int_{|\eta|=1}[\psi_g(t\eta)+\psi_g(-t\eta)-2f(g)]\cdot F(t\eta)D(\eta)\,d\sigma(\eta)\right\}$$

$$t^{m+l-1}\varphi_{n,a}(Rt)\,dt\equiv\sum_{j=0}^{r-1}\zeta_R^{(j)}(f)+\zeta_R(f),$$

其中$C_j(\eta)$及$F(t\eta)$满足

$$|C_j(\eta)|\leqslant C,\quad C_j(-\eta)=(-1)^mC_j(\eta), \tag{11}$$

$$|F(t\eta)|\leqslant Ct^{m+2r},\quad t\leqslant1. \tag{12}$$

由(5)、(9)、(12), 利用引理1及引理3可得

$$\|\zeta_R(f)\|_P \leqslant \begin{cases} CR^{-2r-\gamma}\log R, & a = 2r+\gamma < 2r+2, \\ C(R^{-2r-\gamma}+R^{-a}), & \text{其他}a. \end{cases} \tag{13}$$

令 $\Phi_{g,j}(t) = \int_{|\eta|=1}[\psi_g(t\eta)+\psi_g(-t\eta)-2f(g)]C_j(\eta)D(\eta)\mathrm{d}\sigma(\eta)$. 由于$f\in H_P^{2r+\gamma}(\mathrm{G})$，所以可得 $\Phi_{g,j}(0)=0$，$\Phi_{g,j}^{(2k-1)}(0)=0$，$k=1,2,\cdots,r$. 设$a>2r$，则由引理4得 $\Phi_{g,j}^{(k-1)}(0)H_R^{(2j)}(0)=0$，$k=1,2,\cdots,2r+1-2j$. 由此，对$\zeta_R^{(j)}(f)$ $(j\neq 0)$分部积分$2r+1-2j$. 由此，对

$$\zeta_R^{(j)}(f) = \sum_{k=1}^{2r+1-2j}\frac{C_0}{2}R^{-2j-k+1}\Phi_{g,j}^{(k-1)}(1)H_K^{(2j)}(R) - \frac{C_0}{2}R^{-2r-1}\int_0^R\Phi_{g,j}^{(2r+1-2j)}\left(\frac{t}{R}\right)H_{2r+1-2j}^{(2j)}(t)\mathrm{d}t \equiv \sum_{k=1}^{2r+1-2j}\xi_R^{(k)}+\xi_R.$$

由$f\in H_P^{2r+\gamma}(\mathrm{G})$，利用引理3及Lagrange中值公式可得

$$\|\Phi_{g,j}^{(k)}(t)\|_P \leqslant C\sum_{j_1,\cdots,j_k=1}^{n}\|X_{j_1}\cdots X_{j_k}f\|_p,\quad k=1,2,\cdots,2r-1, \tag{14}$$

$$\|\Phi_{g,j}^{(2k-1)}(t)\|_P < C\sum_{j_1,\cdots,j_{2k}=1}^{n}\|X_{j_1}\cdots X_{j_{2k}}f\|_P t,\quad t>0,\ k=1,\cdots,r. \tag{15}$$

由(14)及引理4得 $\|\xi_R^{(k)}\|_P\leqslant CR^{-a}$，$k=1,2,\cdots,2r+1-2j$. 与引理4的证明类似，利用引理1可得当$a$为偶数时有 $|H_j^{(k)}(t)|\leqslant Ct^{-a+j+k-2}$，$t>1$. 由此及(15)，利用引理4可得 $\|\xi_R\|_P\leqslant C(R^{-a}+R^{-2r-2})$. 因此当$a>2r$ 时有

$$\|\zeta_R^{(j)}(f)\|_P\leqslant C(R^{-a}+R^{-2r-2}),\quad j=1,2,\cdots,r-1. \tag{16}$$

若$a\leqslant 2r$，则可取非负整数 s 使$2s<a\leqslant 2s+2\leqslant 2r$. 由引理4，$f\in H_P^{2s+2}(\mathrm{G})$. 于是用$s$代替$r$与上面同样讨论可得

$$\|\zeta_R^{(j)}(f)\|_P\leqslant C(R^{-a}+R^{-2s-2})\leqslant CR^{-a},\quad j=1,2,\cdots,s-1. \tag{17}$$

由于

$$\zeta_R^{(0)}(f) = \frac{i^mC_0}{2}R^n\left\{\int_0^\infty - \int_1^\infty\right\}\left\{\int_{|\eta|=1}[\psi_g(t\eta)+\psi_g(-t\eta)-2f(g)]\cdot D(\eta)^2\mathrm{d}\sigma(\eta)\right\}t^{n-1}\varphi_{n,a}(Rt)\mathrm{d}t \equiv J_R^{(1)}(f)+J_R^{(2)}(f), \tag{18}$$

由(9)及引理1得

$$\|J_R^{(2)}(f)\|_P\leqslant CR^{-a}. \tag{19}$$

由$f\in H_P^{2r+\gamma}(\mathrm{G})$及引理2可得 $\|\varphi_{g,g_1,\eta}^{(2r)}(t)+\varphi_{g,g_1,\eta}^{(2r)}(-t)-2\varphi_{g,g_1,\eta}^{(2r)}(0)\|_P\leqslant Ct^\gamma$. 由此与[1]中类似可得

$$\|J_R^{(1)}(f)\|_P\leqslant\begin{cases} CR^{-2r-\gamma}, & a>2r+\gamma, \\ CR^{-2r-\gamma}\log R, & a=2r+\gamma<2r+2, \\ CR^{-2r-2}, & a=2r+\gamma=2r+2, \\ CR^{-a}, & a<2r+\gamma. \end{cases} \tag{20}$$

由(8)、(10)、(13)、(16)—(20)，定理1得证.

定理2的证明 与定理1的证明类似可得

$$G_R^a(f,g)-f(g)=I_R^a(f)+J_R^a(f);$$

$$J_R^a(f)=\begin{cases}O(R^{-a}), & k<a<k+1;\\ O(R^{-k-1}\log R), & a=k+1\ \text{为奇数},\\ O(R^{-k-1}), & a>k+1\ \text{或}\ a=k+1\ \text{为偶数};\end{cases}$$

$$I_R^a(f)\ =C\int_0^R\left\{\int_{|\eta|=1}[\psi_g(\frac{t}{R}\eta)-f(g)]D(\eta)^2\mathrm{d}\sigma(\eta)\right\}\ t^{n-1}\varphi_{n,a}(t)\mathrm{d}t.$$

令$\psi_{g,R}$为ψ_g的Стеклов 函数

$$\psi_{g,R}(h)=(\frac{R}{2})^l\int_{-R^{-1}}^{R^{-1}}\cdots\int_{-R^{-1}}^{R^{-1}}\psi_g(h+\sum_{j=1}^{l}y_jY_j)\,\mathrm{d}y,$$

$Y_1,\cdots,Y_l$ 为H的一组标准正交基，$\widetilde{\psi_g}(h)=\psi_g(h)-\psi_{g,R}(h)$，$F_g(t)=\int_{|\eta|=1}[\widetilde{\psi_g}(t\eta)-\widetilde{\psi_g}(0)]D(\eta)^2\mathrm{d}\sigma(\eta)$，$F_{g,R}(t)=\int_{|\eta|=1}[\psi_{g,R}(t\eta)-\psi_{g,R}(0)]D(\eta)^2\mathrm{d}\sigma(\eta)$，则

$$I_R^a(f)=C\int_0^R\left\{F_g(\frac{t}{R})+F_{g,R}(\frac{t}{R})\right\}\ \mathrm{d}H_1^{(0)}(t).\tag{21}$$

由于H是可交换Lie代数，故有

$$\exp(h_1+h_2)=\exp h_1\exp h_2,\ \ h_1,h_2\in \mathrm{H}.\tag{22}$$

设$\mathrm{Ad}g(h)=\sum_{j=1}^{l}\beta_j(g,h)Y_j$，$\eta=\sum_{j=1}^{l}\eta_jY_j$，由$\mathrm{Ad}g$的线性可知$\beta_j(g,h)$关于$h$是线性的．由此及(3)、(22)、与引理2的证明类似，可得当$f\in C^k(G)$时有

$$F_{g,R}^{(0)}(t)=\sum_{i_1,\cdots,i_k=1}^{l}\ \ \sum_{j_1,\cdots,j_{k-1}=1}^{l}\int_{|\eta|=1}\eta_{i_1}\cdots\eta_{i_k}D(\eta)^2\cdot\ \left\{\int_{G/T}\beta_{j_1}(g_1,Y_{i_1})\cdots\right.$$
$$\left.\beta_{j_{k-1}}(g_1,Y_{i_{k-1}})[(\frac{R}{2})^l\int_{-R^{-1}}^{R^{-1}}\cdots\int_{-R^{-1}}^{R^{-1}}\frac{\mathrm{d}}{\mathrm{d}y_{i_k}}Y_{j_1}\cdots Y_{j_{k-1}}f(gg_1\exp(t\eta+\sum_{j=1}^{l}y_jY_j)g_1^{-1})\mathrm{d}y]\mathrm{d}\overline{g_1}\right\}\cdot\mathrm{d}\sigma(\eta).\tag{23}$$

由(23)可知，当$f\in C^k(G)$时$F_{g,R}^{(k+1)}$存在且连续，且

$$F_{g,R}^{(2j-1)}(0)=0,\ 2j-1\leqslant k+1,\tag{24}$$
$$|F_{g,R}^{(k+1)}(t)|\leqslant CR\omega(f^{(k)},\ \frac{1}{R}).\tag{25}$$

同样可得

$$F_g^{(2j-1)}(0)=0,\ 2j-1\leqslant k\tag{26}$$
$$|F_g^{(k)}(t)|\leqslant C\omega(f^{(k)},\ \frac{1}{R})\tag{27}$$

由(21)、(24)—(27)，与定理1的证明类似可得

$$|I_R^a(f)|\leqslant\begin{cases}CR^{1-a}\omega(f^{(k)},\ \frac{1}{R}), & k<a<k+1,\\ CR^{-k}\omega(f^{(k)},\ \frac{1}{R})\log R, & a=k+1,\\ CR^{-k}\omega(f^{(k)},\ \frac{1}{R}), & a>k+1\ \text{或}\ a=k+1\ \text{为偶数}.\end{cases}$$

定理2得证．

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Approximation Properties of the Gauss-Weierstrass Type Means of Fourier Series on Compact Lie Groups

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Abstract

Let G be a compact, connected semisimple Lie group and $f \in L^1(G)$. Denote by $G_R^a(f, g)$ the Gauss-Weierstrass type means of Fourier series of f,

$$G_R^a(f, g) = \sum_{\lambda \in \Lambda(G)} e^{-|\lambda+\beta|^a R^{-a}} d_\lambda \chi_\lambda * f(g), \quad a > 0.$$

In this paper, we consider the problem of approximation to the functions of H $H_P^{2r+\gamma}(G)$ by $G_R^a(f, g)$, where

$$H_P^{2r+\gamma}(G) = \left\{ f \in L\ (G) \,\middle|\, \begin{array}{l} X_{i_1} \cdots X_{i_{2r-1}} f \text{ is absolute continuous} \\ \text{and } X_{i_1} \cdots X_{i_{2r}} f \in H_p^\gamma(G),\ i_1, \cdots, i_{2r} \\ = 1, 2, \cdots, n. \end{array} \right.$$

$$H_p^\gamma(G) = \{ f \in L^p(G) \mid \omega_2(f, t; p) = O(t^\gamma),\ t \to 0 \}$$

$X_1, \cdots, X_n$ is an orthonormal basis for L, the Lie algebra of G, $r = 0, 1, 2, \cdots$, $1 \leqslant p \leqslant \infty$, $0 < \gamma < 2$. We establish the following theorem:

Theorem 1. Let $f \in H_P^{2r+\gamma}(G)$, then

$$\|G_R^a(f) - f\|_p = \begin{cases} O(R^{-2r-\gamma}), & a > 2r+\gamma, \\ O(R^{-2r-\gamma} \log R), & a = 2r+\gamma < 2r+2, \ (R \to +\infty) \\ O(R^{-2r-2}), & a = 2r+\gamma = 2s+2, \\ O(R^{-a}), & a < 2r+\gamma. \end{cases}$$

Moreover, we establish the degree of approximation to the functions of $C^K(G)$ by $G_R^a(f)$ and the asymptotic expansions of the quantity $\sup_{f \in \mathrm{Li}\, p_{1\sigma}} \|G_R^a(f) - f\|_\infty$, $0 < \sigma < 1$.