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一个Zygmund定理的推广及应用*

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一、引　　言

设$\varphi(x)$是实数轴R上有定义且以2π为周期的复值函数，若存在常数A使得对任何$x\in$ R和$y\in$R有

$$|\varphi(x)-\varphi(y)|\leq A|x-y|^{\alpha},\quad 0<\alpha\leq 1,$$

则称$\varphi\in\Lambda_{\alpha}$.

设$\varphi(x)$是R上连续函数，若对正数h_1,h_2,只要$h_1-h_2=O(h/\log\frac{1}{h})$，$h=\max(h_1,h_2)$，就有

$$|\varphi(x+h_1)-2\varphi(x)+\varphi(x-h_2)|\leq Ah,\qquad(1)$$

则称$\varphi\in\Lambda_{**}$. 显然，当$h_1=h_2$时，就是Zygmund函数类Λ_*.

关于解析函数和调和函数之境界值函数的光滑性，Zygmund曾证明了下述定理([3]、[1])：

定理A　设B是单位园，$f(z)$在B内解析，则$f(z)$在$\overline{\mathrm{B}}$上连续和$f(e^{i\theta})\in\Lambda_*$当且仅当

$$f''(z)=O(1/(1-|z|))$$

对一切$z\in$B成立.

定理B　设$f(z)=u(z)+iv(z)$在B内解析，$v(z)$是$u(z)$的共轭调和函数. 若$u(z)$在$\overline{\mathrm{B}}$上连续，且$u(e^{i\theta})\in\Lambda_*$，则$v(z)$在$\overline{\mathrm{B}}$上连续，且$v(e^{i\theta})\in\Lambda_*$.

本文将加强他的这些结论，并将所得结果推广到单连通区域上去.

二、Zygmund定理的加强

定理2.1　设$f(z)$在单位园B内解析，则$f(z)$在$\overline{\mathrm{B}}$上连续，且$f(e^{i\theta})\in\Lambda_{**}$当且仅当

$$f''(z)=O(1/(1-|z|))\qquad(2)$$

对一切$z\in$B成立.

证明　由定理A可知，这定理的必要性是显然的. 我们只要证明其充分性，为此，只要对Zygmund的证明作些必要的修改.

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设（2）成立，则 $f'(z)=O\left(\log\frac{1}{1-|z|}\right)=o\left((1-|z|^{a-1}\right)$，$0<a<1$．故 $f(z)$ 在 $\overline{B}$ 上连续．现只要证明 $f(e^{i\theta})\in\Lambda_{**}$．设 $h_1>0$，$h_2>0$，$h=\max(h_1,h_2)$，$h_1-h_2=O(h/\log\frac{1}{h})$．令

$$\Delta^2_{h_1,h_2}\{f(e^{i\theta})\}=f(e^{i(\theta+h_1)})-2f(e^{i\theta})+f(e^{i(\theta-h_2)}),$$

由［1］(p·77)，有

$$\Delta^2_{h_1,h_2}\{f(e^{i\theta})\}=\Delta^2_{h_1,h_2}\{f(e^{i\theta})-f(\rho e^{i\theta})\}+\Delta^2_{h_1,h_2}\{f(\rho e^{i\theta})\},$$

$$\Delta^2_{h_1,h_2}\{f(e^{i\theta})-f(\rho e^{i\theta})\}=\Delta^2_{h_1,h_2}\{(1-\rho)e^{i\theta}f'(\rho e^{i\theta})\}$$
$$+\Delta^2_{h_1,h_2}\{e^{2i\theta}\int_\rho^1(1-r)f''(re^{i\theta})\mathrm{d}r\}.$$

设 $1-\rho=h$，由条件（2），易知

$$\left|\Delta^2_{h_1,h_2}\{e^{2i\theta}\int_\rho^1(1-r)f''(re^{i\theta})\mathrm{d}r\}\right|\le\int_\rho^1(1-r)\left|f''(re^{i(\theta+h_1)})\right|\mathrm{d}r$$
$$+2\int_\rho^1(1-r)\left|f''(re^{i\theta})\right|\mathrm{d}r+\int_\rho^1(1-r)\left|f''(re^{i(\theta-h_2)})\right|\mathrm{d}r$$
$$=O(h).$$

由 $f'(z)=O(\log\frac{1}{1-|z|})$ 和不等式 $|e^a-e^b|\le e^{|a|+|b|}\cdot|a-b|$，有

$$\left|\Delta^2_{h_1,h_2}\{e^{i\theta}f'(\rho e^{i\theta})\}\right|\le\left|(e^{i(\theta-h_1)}-e^{i\theta})f'(\rho e^{i(\theta+h)})\right|$$
$$+\left|e^{i\theta}(f'(\rho e^{i(\theta+h_1)})-f'(\rho e^{i\theta}))\right|+\left|(e^{i\theta}-e^{i(\theta-h_2)})f''(\rho e^{i\theta})\right|$$
$$+\left|e^{i(\theta-h_2)}(f'(\rho e^{i\theta})-f'(\rho e^{i(\theta-h_2)}))\right|$$
$$\le O(h\log\frac{1}{h})+\int_0^{h_1}\left|f''(\rho e^{i(\theta+t)})\right|\mathrm{d}t+\int_0^{h_2}\left|\rho e^{i(\theta-h_2+t)})\right|\mathrm{d}t$$
$$=O(1).$$

因此 $\Delta^2_{h_1,h_2}\{f(e^{i\theta})-f(\rho e^{i\theta})\}=O(h)$．

最后，

$$\Delta^2_{h_1,h_2}\{f(\rho e^{i\theta})\}=i\rho\{\int_0^{h_1}f'(\rho e^{i(\theta+t)})e^{i(\theta+t)}\mathrm{d}t$$
$$-\int_0^{h_2}e^{i(t+\theta-h_2)}f'(\rho e^{i(t+\theta-h_2)})\mathrm{d}t\}$$
$$=i\rho\{\int_0^{h_1}[e^{i(\theta+t)}f'(\rho e^{i(\theta+t)})-e^{i(t+\theta-h_2)}f'(\rho e^{i(t+\theta-h_2)})]\mathrm{d}t$$
$$+\int_{h_2}^{h_1}e^{i(t+\theta-h_2)}f'(\rho e^{i(t+\theta-h_2)})\mathrm{d}t,$$

$$f'(\rho e^{i(\theta+t)})e^{i(\theta+t)}-f'(\rho e^{i(\theta+t-h_2)})e^{i(\theta-h_2+t)}$$
$$=\int_{t+\theta-h_2}^{t+\theta}[f''(\rho e^{i\tau})+f'(\rho e^{i\tau})]i\rho e^{i\tau}\mathrm{d}\tau$$
$$=O\left(\frac{h_2}{1-\rho}\right)+O\left(h_2\log\frac{1}{1-\rho}\right)=O(1).$$

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于是，注意到$h_1-h_2=O(h/\log\frac{1}{h})$，我们有

$$\Delta^2_{h_1,h_2}\{f(\rho e^{i\theta})\}=O(h)+O(|h_1-h_2|\log\frac{1}{h})=O(h).$$

由于上述$O(1)$都是关于θ是均匀的．故$f(e^{i\theta})\in\Lambda_{**}$ 定理证毕．

推论2.2 设$f(z)=u(z)+iv(z)$在B内解析，$v(z)$是$u(z)$的共轭调和函数，又若$u(z)$在B上连续，且$u(e^{i\theta})\in\Lambda_{**}$，则$v(z)$在B上连续，并且$v(e^{i\theta})\in\Lambda_{**}$.

证明 由文献[1]，(p·83)关于Λ_*的保持性的证明可知，只要用定理2.1代替Zygmund定理，即知这推理是真实的．

三、单连通区域上的Zygmund 定理及其应用

设D是复平面上单连通区域，至少有二个境界点，设$w=g(z)$是D→B上任一共形映照，$z=\varphi(w)$是其逆函数．设ρ_D是D中曲率为 −4 的双曲度量，于是

$$\rho_D(z)=|g'(z)|/(1-|g(z)|^2),$$
$$\rho_D(\varphi(w))|\varphi'(w)|=\rho_B(w)=1/(1-|w|^2). \tag{3}$$

定理3.1 设$f(z)$在D内解析，又D的境界是光滑的若当闭曲线，且$\arg\varphi'(e^{i\theta})\in\Lambda_{**}(\partial B)$，则$f(z)$在D上连续和$f(\zeta)\in\Lambda_{**}(\partial D)$，即对正数$\eta_1,\eta_2,\eta=\max(\eta_1,\eta_2)$，若$\eta_1-\eta_2=O(\eta\ \log\frac{1}{\eta})$ 则有

$$|f(\zeta(s+\eta_1))-2f(\zeta(s))+2f(\zeta(s-\eta_2))|\le A\eta,$$

的充要条件是

$$f''(z)=O(\mathrm{dist}(z,\partial D)^{-1}=O(\rho_D(z)) \tag{4}$$

其中$\zeta(s)$表示∂D关于其弧长的参数表示式，$\mathrm{dist}(z,\partial D)$表示$z\in D$到$\partial D$的距离．

在证明这个定理之前，我们需要下述

引理3.2 设区域D的境界曲线是一光滑的若当曲线，$z=\varphi(w)$：B→D是任一共形映照，其边界对应为$\zeta(s)=\varphi(e^{i\theta})$，$\zeta(s+\eta_1)=\varphi(e^{i(\theta+h_1)})$，$\zeta(s-\eta_2)=\varphi(e^{i(\theta-h_2)})$，若$\arg\varphi'(e^{i\theta})\in\Lambda_{**}$，则

$$h_1>0,\ h_2>0,\ h_1-h_2=O\left(h/\log\frac{1}{h}\right),\ h=\max(h_1,h_2) \tag{5}$$

当且仅当

$$\eta_1>0,\ \eta_2>0,\ \eta_1-\eta_2=O\left(\eta/\log\frac{1}{\eta}\right),\ \eta=\max(\eta_1,\eta_2) \tag{6}$$

时成立．

证明 若(5)成立．由于∂D是一光滑的若当曲线，则$\arg\varphi'(w)$到$\overline{B}$上有一连续扩张([2]，p·295)．又因$\arg\varphi'(e^{i\theta})\in\Lambda_{**}$，据推论2.2，则$\log|\varphi'(w)|$在$\overline{B}$上连续，且$\log|\varphi'(e^{i\theta})|\in\Lambda_{**}$. 因而，$\log\varphi'(w)$在$\overline{B}$上连续，$\log\varphi'(e^{i\theta})\in\Lambda_{**}$，且存在正数$a,b$，使得

$$0>a\le|\varphi'(w)|\le b. \tag{7}$$

由定理2.1，则对$w\in B$有

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$$(\varphi''(w)/\varphi'(w))' = O(1/(1-|w|)) \tag{8}$$

于是，对$w \in B$有

$$\varphi''(w)/\varphi'(w) = O\left(\log\frac{1}{1-|w|}\right). \tag{9}$$

注意到$(\varphi''/\varphi')' = \varphi'''/\varphi' - (\varphi''/\varphi')^2$，由（8）和（9），则有

$$\varphi'''(w)/\varphi'(w) = O(1/(1-|w|)). \tag{10}$$

由于∂D是光滑的若当闭曲线，据[1](p·44)，则

$$\begin{aligned}\eta_1 &= \int_\theta^{\theta+h_1} |\varphi'(e^{it})|\mathrm{d}t = l(\varphi(e^{i(\theta+h_1)}), \varphi(e^{i\theta})) > 0,\\ \eta_2 &= \int_{\theta-h_2}^{\theta} |\varphi'(e^{it})|\mathrm{d}t = l(\varphi(e^{i\theta}), \varphi(e^{i(\theta-h_2)})) > 0,\end{aligned} \tag{11}$$

其中$l(\varphi(e^{i\theta_1}),\varphi(e^{i\theta_2}))$表示$\partial D$上$\varphi(e^{i\theta_1})$和$\varphi(e^{i\theta_2})$之间的弧长.

设

$$\widetilde{\Delta}^2_{h_1,h_2}\{\varphi(e^{i\theta})\} = l(\varphi(e^{i(\theta+h_1)}), \varphi(e^{i\theta})) - l(\varphi(e^{i\theta}), \varphi(e^{i(\theta-h_2)})),$$

$$\widetilde{\Delta}^2_{h_1,h_2}\{\varphi(\rho e^{i\theta})\} = l(\varphi(\rho e^{i(\theta+h_1)}), \varphi(\rho e^{i\theta})) - l(\varphi(\rho e^{i\theta}), \varphi(e^{i(\theta-h_2)})),$$

则

$$\eta_1 - \eta_2 = \widetilde{\Delta}^2_{h_1,h_2}\{\varphi(e^{it})\} = [\widetilde{\Delta}^2_{h_1,h_2}\{\varphi(e^{i\theta})\} - \widetilde{\Delta}^2_{h_1,h_2}\{\varphi(\rho e^{i\theta})\}] + \widetilde{\Delta}^2_{h_1,h_2}\{\varphi(\rho e^{i\theta})\}. \tag{12}$$

设$1-\rho = h$，则

$$\begin{aligned}&|\widetilde{\Delta}^2_{h_1,h_2}\{\varphi(e^{i\theta})\} - \widetilde{\Delta}^2_{h_1,h_2}\{\varphi(\rho e^{i\theta})\}|\\ &= \left|\int_\theta^{\theta+h_1}(|\varphi'(e^{it})| - \rho|\phi'(\rho e^{it})|)\,\mathrm{d}t - \int_{\theta-h_2}^{\theta}(|\varphi'(e^{it})| - \rho|\varphi'(\rho e^{it})|)\,\mathrm{d}t\right|\\ &\le \int_\theta^{\theta+h_1}|\varphi'(e^{it}) - \rho\varphi'(\rho e^{it})|\,\mathrm{d}t + \int_{\theta-h_2}^{\theta}|\varphi'(e^{it}) - \rho|\varphi'(e^{it})|\,\mathrm{d}t.\end{aligned}$$

但是

$$\begin{aligned}\int_\theta^{\theta+h_1}|\varphi'(e^{it}) - \rho\varphi'(\rho e^{it})|\,\mathrm{d}t &\le \int_\theta^{\theta+h_1}|\varphi'(e^{it}) - \varphi'(\rho e^{it})|\,\mathrm{d}t\\ &+ (1-\rho)\int_\theta^{\theta+h_1}|\varphi'(\rho e^{it})|\,\mathrm{d}t.\end{aligned}$$

反复利用部分积分方法，有

$$\varphi'(e^{it}) - \varphi'(\rho e^{it}) = e^{it}(1-\rho)\varphi''(\rho e^{it}) + e^{2it}\int_\rho^1 (1-r)\varphi'''(re^{it})\,\mathrm{d}r,$$

由是，利用（7），（9）和（10），有$\varphi'(e^{it}) - \varphi'(\rho e^{it}) = O\left((1-\rho)\log\frac{1}{1-\rho}\right)$

$$\int_\theta^{\theta+h_1}|\varphi'(e^{it}) - \rho\varphi'(\rho e^{it})|\,\mathrm{d}t = O\left(h^2\log\frac{1}{h}\right).$$

同理，有

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$$\int_{\theta-h_2}^{\theta}|\varphi'(e^{it})-\rho\varphi'(\rho e^{it})|\mathrm{d}t=O\left(h^2\log\frac{1}{h}\right)$$

因此，我们有

$$\widetilde{\Delta}^2_{h_1,h_2}\{\varphi(e^{i\theta})\}-\widetilde{\Delta}^2_{h_1h_2}\{\varphi(\rho e^{it})\}=O\left(h^2\log\frac{1}{h}\right). \tag{13}$$

最后注意到（7）、（9）和假设（5），以及

$$\widetilde{\Delta}^2_{h_1,h_2}\{\varphi(\rho e^{it})\}=\rho\left(\int_0^{h_1}|\varphi'(\rho e^{i(t+\theta)})|\mathrm{d}t-\int_0^{h_2}|\varphi'(e^{i(t+\theta-h_2)})|\mathrm{d}t\right)$$
$$\leq\int_0^{h_1}|\varphi'(\rho e^{i(t+\theta)})-\varphi'(\rho e^{i(t+\theta-h_2)})|\mathrm{d}t+\left|\int_{h_2}^{h_1}|\varphi'(\rho e^{i(t+\theta-h_2)})|\mathrm{d}t\right|,$$
$$|\varphi'(\rho e^{i(t+\theta)})-\varphi'(\rho e^{i(t+\theta-h_2)})|\leq\int_0^{h_2}|\varphi''(\rho e^{i(\tau+t+\theta-h_2)})|\mathrm{d}\tau,$$

我们便有

$$\widetilde{\Delta}^2_{h_1,h_2}\{\varphi(\rho e^{it})\}=O\left(h^2\log\frac{1}{h}\right)+O(|h_1-h_2|)=O\left(h/\log\frac{1}{h}\right). \tag{14}$$

由是，由（13）、（14）和（12），我们有

$$\eta_1-\eta_2=O\left(h/\log\frac{1}{h}\right).$$

但是，由（7），有$ah_j\leq\eta_j\leq b\eta_j$, $j=12$)，于是，

$$ah\leq\eta\leq bh. \tag{15}$$

可见$O\left(h/\log\frac{1}{h}\right)=O\left(\eta/\log\frac{1}{\eta}\right)$，故（6）式成立.

反之，若（6）式成立，令

$$\zeta(\tau)=\varphi(e^{it}),\tau=\int_0^t|\varphi'(e^{i\eta})|\mathrm{d}\eta,\zeta_\rho(\tau)=\varphi(\rho e^{it}),$$

则

$$|\mathrm{d}\zeta(\tau)|=|\varphi'(e^{it})|\mathrm{d}t=\mathrm{d}\tau,\quad|\mathrm{d}\zeta_\rho(\tau)|=\rho|\varphi'(\rho e^{it})/\varphi'(e^{it})|\mathrm{d}\tau. \tag{16}$$

据设$\eta_1>0$，$\eta_2>0$，则

$$h_1=l(g(\zeta(s+\eta_1)),g(\zeta(s)))=\int_s^{s+\eta_1}|g'(\zeta(\tau))|\mathrm{d}\tau>0,$$
$$h_2=l(g(\zeta(s)),g(\zeta(s-\eta_2)))=\int_{s-\eta_2}^{s}|g'(\zeta(\tau))|\mathrm{d}\tau>0.$$

设

$$\widetilde{\Delta}^2_{\eta_1,\eta_2}\{g(\zeta(s))\}=l\big(g(\zeta(s+\eta_1)),g(\zeta(s))\big)-l\big(g(\zeta(s)),g(\zeta(s-\eta_2))\big),$$

$$\widetilde{\Delta}^2_{\eta_1,\eta}\{g(\zeta_\rho(s))\}=l\big(g(\zeta_\rho(s+\eta_1)),g(\zeta_\rho(s))\big)-l\big(g(\zeta_\rho(s)),g(\zeta_\rho(s-\eta_2))\big),$$

则显然，

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$$h_1 - h_2 = [\widetilde{\Delta}^2_{\eta,\eta_2}\{g(\zeta(s))\} - \widetilde{\Delta}^2_{\eta_1,\eta_2}\{g(\zeta_\rho(s))\}] + \widetilde{\Delta}^2_{\eta_1,\eta_2}\{g(\zeta_\rho(s))\}. \tag{17}$$

$$\widetilde{\Delta}^2_{\eta_1,\eta_2}\{g(\zeta(s)) - g(\zeta_\rho(s))\}$$
$$= \int_s^{s+\eta_1}(|g'(\zeta(\tau))| - \rho\left|\frac{\varphi'(\rho e^{it})}{\varphi'(e^{it})}\right||g'(\zeta_\rho(\tau))|)d\tau$$
$$- \int_{s-\eta_2}^{s}(|g'(\zeta(\tau))| - \rho\left|\frac{\varphi'(\rho e^{it})}{\varphi'(e^{it})}\right||g'(\zeta_\rho(\tau))|)d\tau$$
$$\leq \int_s^{s+\eta_1}\left|g'(\zeta(\tau)) - \rho\frac{\varphi'(\rho e^{it})}{\varphi'(e^{it})}g'(\zeta_\rho(\tau))\right|d\tau$$
$$+ \int_{s-\eta_2}^{s}\left|g'(\zeta(\tau)) - \rho\frac{\varphi'(\rho e^{it})}{\varphi'(e^{it})}g'(\zeta_\rho(\tau))\right|d\tau. \tag{18}$$

但是，

$$\int_s^{s+\eta_1}\left|g'(\zeta(\tau)) - \rho\frac{\varphi'(\rho e^{it})}{\varphi'(e^{it})}g'(\zeta_\rho(\tau))\right|d\tau$$
$$\leq \int_s^{s+\eta_1}\left(|g'(\zeta(\tau)) - g'(\zeta_\rho(\tau))| + |g'(\zeta_\rho(\tau))|(1-\rho) + |g'(\zeta_\rho(\tau))| \cdot \left|1 - \frac{\varphi'(\rho e^{it})}{\varphi'(e^{it})}\right|\right)d\tau.$$

设 $1-\rho = \eta = \max(\eta_1, \eta_2)$， 注意到 $|g'(\zeta_\rho(\tau))| = |\varphi'(\rho e^{it})|^{-1} \leq a^{-1}$，易知

$$I_2 = \int_s^{s+\eta_1}(1-\rho)|g'(\zeta_\rho(\tau))|d\tau = O(\eta^2). \tag{19}$$

$$I_3 = \int_s^{s+\eta_1}|g'(\zeta_\rho(\tau))|\left|1 - \frac{\varphi'(\rho e^{it})}{\varphi'(e^{it})}\right|d\tau$$
$$\leq \frac{1}{a}\int_s^{s+\eta_1}|\varphi'(e^{it}) - \varphi'(\rho e^{it})|dt$$
$$\leq \frac{1}{a}\int_s^{s+\eta_1}[(1-\rho)|\varphi''(\rho e^{it})| + \int_\rho^1(1-r)\varphi'''(re^{it})dr]dt.$$

于是，利用（7）、（9）和（10）便得

$$I_3 = O(\eta^2\log\frac{1}{h}) + O(\eta^2) = O(\eta^2\log\frac{1}{\eta}). \tag{20}$$

现估计

$$I_1 = \int_s^{s+\eta_1}|g'(\zeta(\tau)) - g'(\zeta_\rho(\tau))|d\tau.$$

反复利用部分积分法，并注意到$\zeta_r(\tau) = \varphi(re^{it})$，则

$$g'(\zeta(\tau)) - g'(\zeta_\rho(\tau)) = \int_\rho^1 g''(\zeta_r(\tau))\varphi'(re^{it})e^{it}dr$$
$$= e^{it}\{g''(\zeta(\tau))\varphi'(e^{it}) - \rho g''(\zeta_\rho(\tau))\varphi'(\rho e^{it}) - \int_\rho^1 r d(g''(\zeta_r(\tau))\varphi'(re^{it}))\}$$
$$= e^{it}\{(1-\rho)g''(\zeta_\rho(\tau))\varphi'(\rho e^{it}) + e^{it}\int_\rho^1(1-r)d[g''(\zeta_r(\tau))\varphi'(re^{it})]\}. \tag{21}$$

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但是，由于$1=g'(\zeta_r(\tau))\varphi'(re^{it})$，$\zeta_r(\tau)=\varphi(re^{it})$，易知

$$g''(\zeta_r(\tau))\varphi'(re^{it})=-g'(\zeta_r(\tau))\frac{\varphi''(re^{it})}{\varphi(re^{it})}$$

$$\mathrm{d}(g''(\zeta_r(\tau)),\varphi'(re^{it}))=-\Big[\Big(\frac{\varphi''(re^{it})}{\varphi'(re^{it})}\Big)'g'(\zeta_r(\tau))+\Big(\frac{\varphi''(re^{it})}{\varphi'(re^{it})}\Big)g''(\zeta_r(\tau)\varphi'(re^{it})\Big]e^{it}\mathrm{d}r.$$

由（7）、（8）和（9），则得

$$\begin{cases} g''(\zeta_r(\tau))\varphi(re^{it})=O\Big(\log\frac{1}{-r}\Big) \\ \mathrm{d}(g''(\zeta_r(\tau)),\varphi'(re^{it}))/\mathrm{d}r=O\Big(\frac{1}{1-r}\Big). \end{cases} \tag{22}$$

由是，由（21）和（22），则得

$$g'(\zeta(\tau))-g'(\zeta_\rho(\tau))=O\Big(\eta\log\frac{1}{\eta}\Big).$$

于是我们获得

$$I_1=O(\eta^2\log\frac{1}{\eta}). \tag{23}$$

由是，由（19）、（20）和（23），逐得

$$\int_s^{s+\eta_1}\Big|g'(\zeta(\tau))-\rho\frac{\varphi'(\rho e^{it})}{\varphi'(e^{it})}g'(\zeta_\rho(\tau))\Big|\mathrm{d}\tau=O(\eta^2\log\frac{1}{\eta}).$$

同理，有

$$\int_{s-\eta_2}^{s}\Big|g'(\zeta(\tau))-\rho\frac{\varphi'(\rho e^{it})}{\varphi'(e^{it})}g'(\zeta_\rho(\tau))\Big|\mathrm{d}\tau=O(\eta^2\log\frac{1}{\eta}).$$

由是，由（18），便得

$$\widetilde{\Delta}^2_{\eta_1,\eta_2}\{g(\zeta(\tau))-g(\zeta_\rho(\tau))\}=O(\eta^2\log\frac{1}{\eta}). \tag{24}$$

现在，我们来讨论$\widetilde{\Delta}^2_{\eta_1,\eta_2}\{g(\zeta_\rho(\tau))\}$，据定义，

$$\begin{aligned}\widetilde{\Delta}^2_{\eta_1,\eta_2}\{g(\zeta_\rho(\tau))\}&=\int_s^{s+\eta_1}|g'(\zeta_\rho(\tau))\mathrm{d}\zeta_\rho(\tau)|-\int_{s-\eta_2}^{s}|g'(\zeta_\rho(\tau))\mathrm{d}\zeta_\rho(\tau)|\\ &=\rho\Big(\int_0^{\eta_1}\Big|g'(\zeta_\rho(s+\tau))\frac{\varphi'(\rho e^{i(\theta+t)})}{\varphi'(e^{i(\theta+t)})}\Big|\mathrm{d}\tau\\ &\quad-\int_0^{\eta_2}\Big|g'(\zeta_\rho(\tau+s-\eta_2))\frac{\varphi'(\rho e^{i(t+\theta-h_2)})}{\varphi'(e^{i(t+\theta-h_2)})}\Big|\mathrm{d}\tau\Big)\\ &\le\int_0^{\eta_1}\Big|g'(\zeta_\rho(s+\tau))\frac{\varphi'(\rho e^{i(\theta+t)})}{\varphi'(e^{i(\theta+t)})}-g'(\zeta_\rho(\tau+s-\eta_2))\frac{\varphi'(\rho e^{i(t+\theta-h_2)})}{\varphi'(e^{i(t+\theta-h_2)})}\Big|\mathrm{d}\tau\\ &\quad+\Big|\int_{\eta_2}^{\eta_1}\Big|g'(\zeta_\rho(\tau+s-\eta_2))\frac{\varphi'(\rho e^{i(t+\theta-h_2)})}{\varphi'(e^{i(t+\theta-h_2)})}\Big|\mathrm{d}\tau.\end{aligned} \tag{25}$$

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注意到$1=g'(\zeta_\rho(\tau))\varphi'(\rho e^{it})$，由（7），便得

$$I_4=\int_{\eta_2}^{\eta_1}\left|g'(\zeta_\rho(\tau+s-\eta_2)\frac{\varphi'(\rho e^{i(t+\theta-h_2)})}{\varphi'(e^{i(t+\theta-h_2)})}\right|d\tau=O(\eta_1-\eta_2)$$

$$=O(\eta/\log\frac{1}{\eta}). \tag{26}$$

关于

$$I_5=\int_0^{\eta_1}\left|g'(\zeta_\rho(s+\tau))\frac{\varphi'(\rho e^{i(\theta+t)})}{\varphi'(e^{i(\theta+t)})}-g'(\zeta_\rho(\tau+s-\eta_2)\frac{\varphi'(\rho e^{i(t+\theta-h_2)})}{\varphi'(e^{i(t+\theta-h_2)})}\right|d\tau$$

$$\leq\int_0^{\eta_1}\left\{\left|g'(\zeta_\rho(s+\tau))\left(1-\frac{\varphi'(\rho e^{i(\theta+t)})}{\varphi'(e^{i(\theta+t)})}\right)\right|+\left|g'(\zeta_\rho(\tau+s)-g'(\zeta_\rho(\tau+s-\eta_2)\right|\right.$$

$$\left.+\left|g'(\zeta_\rho(\tau+s-\eta_2))\left(1-\frac{\varphi'(\rho e^{i(t+\theta-h_2)})}{\varphi'(e^{i(t+\theta-h_2)})}\right)\right|\right\}d\tau$$

$$=\int_0^{\eta_1}(I_{51}+I_{52}+I_{53})d\tau.$$

注意到$\varphi'(e^{it})-\varphi'(\rho e^{it})=O\left((1-\rho)\log\frac{1}{1-\rho}\right)$和不等式（7），便得

$$I_{51}=\left|g'(\zeta_\rho(s+\tau))\frac{\varphi'(e^{i(\theta+t)})-\varphi'(\rho e^{i(\theta+t)})}{\varphi'(e^{i(\theta+t)})}\right|=O(\eta\log\frac{1}{\eta}),$$

$$I_{53}=\left|g'(\zeta_\rho(\tau+s-\eta_2))\left(1-\frac{\varphi'(\rho e^{i(t+\theta-h_2)})}{\varphi'(e^{i(t-\theta-h_2)})}\right)\right|=O(\eta\log\frac{1}{\eta}).$$

由于

$$I_{52}=\left|g'(\zeta_\rho(\tau+s))-g'(\zeta_\rho(\tau+s-\eta_2))\right|=\left|g'(\varphi(\rho e^{i(t+\theta)}))-g'(\varphi(\rho e^{i(t+\theta-h_2)}))\right|$$

$$=\left|\int_{t+\theta-h_2}^{t+\theta}g''(\varphi(\rho e^{i\tau}))\varphi'(\rho e^{i\tau})\rho d\tau\right|,$$

由不等式（22）和（15），便得

$$I_{52}=O(h_2\log\frac{1}{\eta})=O(\eta\log\frac{1}{\eta}).$$

综上所述，我们有$I_5=O(\eta^2\log\frac{1}{\eta})$. 由是，由（25）和（26），便有

$$\widetilde{\Delta}^2_{\eta_1,\eta_2}\{g(\zeta_\rho(\tau))\}=O(\eta/\log\frac{1}{\eta}). \tag{27}$$

联立（17）、（24）和（27），以及（15），便得

$$h_1-h_2=O(h/\log\frac{1}{h}).$$

故（5）式成立．引理证毕．

定理3.1的证明 先证必要性，令$F(w)=f\circ\varphi(w)$，则$F(w)$在B内解析，在$\overline{\mathrm{B}}$上连续，对正数h_1,h_2，若$h_1-h_2=O(h/\log\frac{1}{h})$，$h=\max(h_1,h_2)$，据引理，则$\eta_1,\eta_2>0$，$\eta_1-\eta_2=O(\eta/\log\frac{1}{\eta})$，$\eta=\max(\eta_1,\eta_2)$，由于$f(\zeta(s))\in\Lambda_{**}(\partial D)$，于是有

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$$|F(e^{i(\theta+h_1)})-2F(e^{i\theta})+F(e^{i(\theta-h_2)})|$$
$$=|f(\zeta(s+\eta_1))-2f(\zeta(s))-f(\zeta(s-\eta_2))|$$
$$\leq A\eta\leq Abh.$$

故$F(e^{i\theta})\in\Lambda_{**}$,据定理2.1,则对一切$w\in B$有

$$F''(w)=O((1-|w|)^{-1}),\quad F'(w)=O(\log(1-|w|)^{-1}).$$

由是,注意到关系式$F''(w)=f''(z)(\varphi'(w))^2+f'(z)\varphi''(w)$ 和不等式(7)、(9),便有

$$f''(z)=(\varphi'(w))^{-2}[F''(w)-F'(w)\frac{\varphi''(w)}{\varphi'(w)}]=O(1-|w|)^{-1})$$
$$=O(|g'(z)|/(1-|g(z)|^2))=O(\rho_D(z)).$$

由于

$$\frac{1}{4}\mathrm{dist}^{-1}(z,\partial D)\leq\rho_D(z)\leq\mathrm{dist}^{-1}(z,\partial D),\tag{28}$$

故$f''(z)=O(\rho_D(z))=O(\mathrm{dist}^{-1}(z,\partial D))$,这就是我们所要证的(4)

反之,若(4)式成立,则对$g(z)=w$,

$$f''(z)=O(\rho_D(z))=O((1-|w|)^{-1}),f'(z)=O(\log\frac{1}{1-|w|}).$$

于是我们有 $F''(w)=O((1-|w|)^{-1})$,据定理2·1,则$F(w)$ 在$\bar{B}$上连续,且$F(e^{i\theta})\in\Lambda_{**}(\partial B)$.从而,$f(z)$ 在$\bar{D}$上连续,并且$f(\zeta)\in\Lambda_{**}(\partial D)$,事实上,对正数$\eta_1,\eta_2$,若$\eta_1-\eta_2=O(\eta/\log\frac{1}{\eta})$,$\eta=\max(\eta_1,\eta_2)$.据引理,则对应的$h_1>0$,$h_2>0$,且$h_1-h_2=O(h\ \log\frac{1}{h})$,$h=\max(h_1,h_2)$.于是,由于$F(e^{i\theta})\in\Lambda_{**}(\partial B)$,则

$$|f(\zeta(s-\eta_1))-2f(\zeta(s))-f(\zeta(s-\eta_2))|$$
$$=|F(e^{i(\theta+h_1)}(-2|F(e^{i\theta})|+F(e^{i(\theta-h_2)})|\leq Ah\leq(A/a)\eta.$$

故$f(\zeta)\in\Lambda_{**}(\partial D)$.定理证毕.

类似地,利用引理3.2和推论2.2,不难证明下述

定理3.3 设区域D的境界∂D是光滑的若当闭曲线,$z=\varphi(w)$:$B\to D$是任一共形映照,又设$\arg\varphi'(e^{i\theta})\in\Lambda_{**}(\partial B)$.设函数$f(z)=u(z)+iv(z)$在D内解析,$v(z)$是$u(z)$的共轭调和函数,若$u(z)$在$\bar{D}$上连续,在境界上$u(\zeta)\in\Lambda_{**}(\partial D)$,则$v(z)$在$\bar{D}$上连续,且$v(\zeta)\in\Lambda_{**}(\partial D)$.

定理3.4 设区域D的境界是一光滑的闭若当曲线,$z=\varphi(w)$:$B\to D$是任一共形映照,又$\arg\varphi'(e^{i\theta})\in\Lambda_{**}(\partial B)$.设$f(z)$在D内解析$f'(z)\neq0$.若$\arg f'(z)$在$\bar{D}$上连续,且$\arg f'(\zeta)\in\Lambda_{**}(\partial D)$,则$\log f'(z)$在$\bar{D}$上连续,$\log f'(\zeta)\in\Lambda_{**}(\partial D)$,此外

$$f''(z)/f'(z)=O(\log\rho_D(z))=O(\log\mathrm{dist}^{-1}(z,\partial D)),\tag{29}$$

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$$\{f, z\} = O(\rho_D(z)) = O(\mathrm{dist}^{-1}(z, \partial D)). \tag{30}$$

其中 $\{f, z\} = (f''/f')' - \frac{1}{2}(f''/f')^2$ 是 f 在点 z 的Schwarg导数.

这定理改进了作者［1］的有关结果.

证明 设 $h(z) = \frac{1}{i}\log f'(z)$, $\log 1 = 0$，则 $h(z)$ 在D内解析，其实部满足定理**3.3**的条件因此，$\log|f'(z)|$ 在 $\overline{D}$ 上连续，$\log|f'(\zeta)| \in \Lambda_{**}(\partial D)$，故 $\log f'(z)$ 在 $\overline{D}$ 上连续，$\log f'(\zeta) \in \Lambda_{**}(\partial D)$. 由是，据定理3.1则

$$(f''(z)/f'(z))' = O(\rho_D(z)). \tag{31}$$

设 $z_0 = \varphi(0)$, $w = g(z)$ 是 $z = \varphi(w)$ 的逆函数，对 $(f''/f')'$ 沿着直线段 $\overline{Ow}$ 在 $z = \varphi(w)$ 的象曲线积分，利用不等式（7）和（31），有

$$\left|\frac{f''(z)}{f'(z)} - \frac{f''(z_0)}{f'(z_0)}\right| \le \int_0^{|w|} \left|\left(\frac{f''(\varphi(w))}{f'(\varphi(w))}\right)'\right| |\varphi'(w)| d|w|$$

$$\le c_1 \int_0^{|w|} \frac{d|w|}{(1-|w|^2)} = c_2 \log\frac{1+|w|}{1-|w|} \le c_3 \log\rho_D(z).$$

由是，即知（29）是成立的．于是，由 $\{f, z\}$ 与（f''/f'）的关系式和（29）、(31)，便可获得（30）．定理证毕.

定理3.5 设区域D的境界是光滑的若当闭曲线，$z = \varphi(w)$: B → D是任一共形映照，又 $\arg\varphi'(e^{i\theta}) \in \Lambda_{**}(\partial B)$. 设 $f(z)$ 在D内解析，若

$$(f''(z)/f'(z))' = O(\rho_D(z)) \tag{32}$$

对一切 $z \in D$ 成立，则 $\partial f(D)$ 是一光滑曲线，$\log f'(z)$ 在 $\overline{D}$ 上连续且

$$\log f'(\zeta) \in \Lambda_{**}(\partial D).$$

这定理改进了作者［4］的有关结果.

证明 由条件（32），显然 $f'(z) \neq 0$，$z \in D$. 设 $h(z) = \log f'(z)$，则 $h(z)$ 在D内解析，且对一切 $z \in D$,

$$h''(z) = O(\rho_D(z)).$$

据定理3.1，则 $h(z)$ 在 $\overline{D}$ 上连续，$h(\zeta) \in \Lambda_{**}(\partial D)$. 因此，$\arg f'(z)$ 在 $\overline{D}$ 上连续，当然在 ∂D 上连续，故 $\partial f(D)$ 是一光滑曲线．定理证毕.

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