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Factorization of Some Meromorphic Functions of Infinite Order

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I. Introduction and Main Results

A Meromorphic function $F(z)$ is said to have a non-trivial factorization with left factor f and right factor g, if

(1) $$F(z)=f(g(z)),$$

where f is a non-linear meromorphic function and g is a non-linear entire function (g may be meromorphic when f is rational). $F(z)$ is said to be prime, if it has no non-trivial factorizations, i.e. if (1) implies that either f or g is linear. $F(z)$ is said to be pseudo-prime, if (1) implies that one of f and g is not transcendental. Two factorizations $F=f_1(g_1)$ and $F=f_2(g_2)$ are said to be equivalent, if there are linear transfomations λ_1, λ_2 and λ_3 such that

$$f_2=\lambda_1\circ f_1\circ\lambda_2 \quad \text{and} \quad g_2=\lambda_2^{-1}\circ g_1\circ\lambda_3.$$

We shall consider two factorizations of a meromorphic function as the same one if they are equivalent.

Ozawa[2] investigated the pseudo-primality of some classes of entire functions as follows:

(A) $F(z)=P(z)e^{H(z)}$, where $P(z)$ is a non-constant polynomial, and $H(z)$ is a non-constant entire function of order<1.

(B) $F(z)=\int_0^z P(z)e^{H(z)}dz$, where P and H are the same as in (A).

(C) $F(z)=P(z)\exp(e^z)$ with a non-constant polynomial $P(z)$.

(D) $F(z)=\int_0^z P(z)\exp(e^z)dz$ with a non-constant polynomial $P(z)$.

We shall in this note pursue the subject of the investigation further to discuss the primality of the above functions with the more general forms. The main results are contained in the following theorems.

Theorem I Let $F(z)=Q(z)e^{H(z)}$, where Q and H satisfy one of the following hypotheses:

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(Ⅰ) $Q(z)$ is a rational function such that none of Q and $\frac{1}{Q}$ is a polynomial, and $H(z)$ is a non-constant entire function;

(Ⅱ) $Q(z)$ is a non-constant rational function, and $H(z)$ is a non-constant entire function of order $\rho(H)<1$.

Then $F(z)$ is prime, unless (i) $Q(z)=\beta(z)^n$, where $\beta(z)$ is rational, and $n\geq 2$ is an integer; or (ii) $Q(z)$ and $H(z)$ have a common rightfactor $a(z)$, i.e. there exist $q(\zeta)$, $h(\zeta)$ and $a(z)$ such that

$$Q(z)=q(a(z)) \text{ and } H(z)=h(a(z)),$$

where $q(\zeta)$ is rational, $h(\zeta)$ is entire, and $a(z)$ is a polynomial of degree $\geq$ 2.

Remark In [5] the author has obtained the same result as theorem 1 for the function $F(z)=Q(z)e^{H(z)}$ with a polynomial $H(z)$ (so that F is of finite order). But when $H(z)$ is transcendental, we have to use the different method.

Theorem 2 Let $F(z)$ be a meromorphic function satisfying $F'(z)=Q(z)e^{H(z)}$, where $Q(z)\not\equiv 0$ is a rational function, and $H(z)$ is a transcendental entire function of order <1, Then $F(z)$ is prime, unless there exist rational function $q(\zeta)$, polynomial $a(z)$ of degree ≥ 2, and entire function $h(\zeta)$, such that

$$Q(z)=a'(z)\,q(a(z)) \text{ and } H(z)=h(a(z)).$$

Theorem 3 Let $F(z)=Q(z)\exp(e^z)$, where $Q(z)$ is a non-constant rational function. Then $F(z)$ is prime, unless $Q(z)=\beta(z)^n$ with a rational function $\beta(z)$ and $n\geq 2$.

Theorem 4 Let $F(z)$ be a meron orphic function satisfying $F'(z)=Q(z)\exp(e^z)$ with a rational function $Q(z)$. Then $F(z)$ is prime.

2. Proofs of Theorems

In proving our theorems we need some lemmas.

Lemma 1[1] Let $a_j(z)$ $(j=0, 1, \cdots, n)$ be meromorphic functions of order $\leq\rho$, and $g_j(z)$ $(j=1, \cdots, n)$ be entire functions and $g_j(z)-g_k(z)$ $(j\neq k)$ transcendental entire functions or polynomials of degree $>\rho$. Then the identity relation

$$\sum_{j=1}^{n} a_j(z)e^{g_j(z)}=a_0(z)$$

holds only when $a_0(z)=a_1(z)=\cdots=a_n(z)\equiv 0$.

Lemma 2[3] Let f and g be transcendental entire functions. If f is of positive order, then $f(g)$ is of infinite order.

Lemma 3 All possible non-trivial factorizations for e^z of the form $e^z=$

$f((g))$ are the following (in the sense of equivalency)

$$f(\zeta)=\zeta^{n}\ (|n|\geq 2),\quad g(z)=e^{z/n}.$$

Proof of Theorem 1 We just prove this theorem under hypothesis (I); the proof under (II) is quite similar. Now we discuss 3 cases.

a) $F=f(g)$, where $f(\zeta)$ is meromorphic, and g entire, both transcendental. Since F has only finitely many zeros and poles, we can see that $g(z)$ must have one finite Picard exceptional values, which may be assumed to be $\zeta=0$ (by a linear transformation, which doesn't affect the factorization of $F(z)$ in the sense of equivalency), and $f(\zeta)$ has its zero or pole only at $\zeta=0$ and $\zeta=\infty$. Thus

$$f(\zeta)=\zeta^{n}e^{\varphi(\zeta)} \text{ and } g(z)=p(z)e^{\psi(z)},$$

where $n\neq 0$ is an integer, $p(z)\not\equiv 0$ is a polynomial, and φ, ψ are non-constant entire functions. We obtain

$$Q(z)\exp\{H(z)\}=p(z)^{n}\exp\{n\psi(z)+\varphi(p(z)e^{\psi(z)})\}.$$

By lemma 1, we deduce $Q(z)=cp(z)^{n}$, which implies that Q or $\frac{1}{Q}$ is a polynomial. Thus we get a contradiction.

b) $F=R(g)$, where $R(\zeta)$ is a rational function of degree ≥ 2, and g meromorphic. By the same reasoning as stated in case a), we may write

$$R(\zeta)=c\zeta^{n},\quad g(z)=\beta_1(z)e^{h(z)},$$

where c is a constant, n is an integer with $|n|\geq 2$, $\beta_1(z)$ is a rational function, and $h(z)$ is a non-constant entire function. Therefore,

$$Q(z)e^{H(z)}=c\beta_1(z)^{n}e^{nh(z)}.$$

which implies $Q(z)=\beta(z)^{n}$ with a rational function $\beta(z)=c_1\beta_1(z)$.

c) $F=f(a)$, where $f(\zeta)$ is meromorphic, and $a(z)$ is a polynomial of degree ≥ 2. Since $f(\zeta)$ can only have finitely manv zeros and poles,

$$f(\zeta)=q_1(\zeta)e^{h_1(\zeta)}$$

where $q_1(\zeta)$ is rational, and $h_1(\zeta)$ entire. Hence,

$$Q(z)\exp\{H(z)\}=q_1(a(z))\exp\{h_1(a(z))\},$$

which implies $Q(z)=q(a(z))$ and $H(z)=h(a(z))$, where $q=c_1q_1$ and $h=h_1+c_2$ with constants c_1 and c_2.

Proof of Theorem 2 Let $F=f(g)$. Then

$$F'(z)=f'(g(z))g'(z)=Q(z)e^{H(z)}. \tag{2}$$

a) f is meromorphic, and g is entire, both transcendental. The discussion in this case is quite similar to that of Ozawa's theorem in [2], which should be omitted.

b) f is rational of degree ≥ 2, and g meromorphic. From (2) we know that $g(z)$ has only finitely many poles, and it has exactly one Picard exceptional value, which may be assumed to be $\zeta=0$, and $f'(\zeta)$ has its zeros or poles only at $\zeta=0$ and $\zeta=\infty$. Thus, we have

$$f'(\zeta)=c\zeta^n,\quad g(z)=q(z)e^{h(z)}, \tag{3}$$

where c is a constant, $n\neq 0,\ -1$, $q(z)$ is rational, and $h(z)$ transcendental entire. From (2) and (3) we deduce

$$cq(z)^n[q'(z)+q(z)h'(z)]\exp\{(n+1)h(z)\}=Q(z)e^{H(z)}. \tag{4}$$

Since $h'(z)$ is obviously transcendental of order <1, $(n+1)h(z)-H(z)$ can neither be transcendental (by lemma2) nor a non-constant polynomial (otherwise $h(z)$ would be of order ≥ 1) nor a constant. Therefore, equality (4) is untenable.

c) f is meromorphic, and $g(z)=a(z)$ is a polynomial. Since $f'(\zeta)$ can only have finitely many zeros and poles, we have

$$f'(\zeta)=q_1(\zeta)\exp\{h_1(\zeta)\},$$

where $q_1(\zeta)$ is rational, and $h_1(\zeta)$ transcendental entire. Thus

$$q_1(a(z))a'(z)\exp\{h_1(a(z))\}=Q(z)\exp\{H(z)\}.$$

By lemma 1, we obtain $Q(z)=q(a(z))a'(z)$, $H(z)=h(a(z))$, where $q=c_1q_1$, $h=h_1+c_2$ with constants c_1, c_2.

The proof of theorem 3 is quite similar to that of theorem 1, which is omitted here.

Proof of Theorem 4 Let $F=f(g)$. Then

$$F'(z)=f'(g(z))g'(z)=Q(z)\exp(e^z) \tag{5}$$

a) f is meromorphic, and g entire, both transcendental. By the same argument as in the proof of theorem 1, we obtain

$$f'(\zeta)=\zeta^n e^{\varphi(\zeta)},\quad g(z)=p(z)e^{\psi(z)}, \tag{6}$$

where $n\neq -1$ is an integer. φ and ψ are non-constant entire (actually, $\psi(z)$ must be a polynomial, as is easily seen), and $p(z)\not\equiv 0$ is a polynomial, From (5) and (6) we deduce

$$\exp\{(n+1)\psi(z)+\varphi(p(z)e^{\psi(z)})-e^z\}=\frac{p(z)^n}{Q(z)}[p'(z)+p(z)\psi'(z)].$$

But this equality can not hold (by careful discussing, the details should be omitted).

b) f is rational of degree ≥ 2, and g meromorphic. In this case, we may get

$$f'(\zeta)=c\zeta^n,\quad g(z)=q(z)e^{h(z)},$$

where $n\neq 0$, -1 is an integer, $q(z)\not\equiv 0$ rational, and $h(z)$ entire. It follows that

$$cq(z)^n[q'(z)+q(z)h'(z)]\exp\{(n+1)h(z)\}=Q(z)\exp(e^z).$$

Obviously, $h(z)$ is of order 1, so is $h'(z)$. Hence, we deduce

$$(n+1)h(z)=e^z+az+c_1,\ h'(z)=\frac{e^z}{n+1}+\frac{a}{n+1}\quad(a\neq 0)$$

Therefore,

$$cq(z)^n\left[q'(z)+\frac{q(z)}{n+1}e^z+\frac{a}{n+1}g(z)\right]e^{az+c_1}=Q(z)$$

But this would derive $q(z)=Q(z)=0$ by lemma 1, a contradiction.

c) f is meromorphic, and g is a polynomial of degree≥ 2. Then

$$f'(\zeta)=R(\zeta)e^{\varphi(\zeta)}\text{ and }R(g(z))g'(z)\exp\{\varphi(g(z))\}=Q(z)\exp(e^z),$$

where φ is entire, and $R(\zeta)\not\equiv 0$ is rational. By lemma 1, we would derive

$$e^z=\varphi(g(z))+c. \tag{7}$$

But (by lemma 3) e^z can not have a factorization like the form (7).

References

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