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Finite Element Method for a Class of Nonlinear Problems (Ⅱ)——Applications*

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Continuing the work in [1], we apply the results in [1] to the finite element methods for the Navier-Stokes' equations and the von Karman equations, and show the convergence of some conforming elements, nonconforming elements and quasi-conforming elements, which are convergent in the cases of the linear problems.

Ⅰ. Finite Element Spaces

Let Ω be a polyhedroid domain in R^n. Denote β the multi-index with $|\beta|=\sum_{i=1}^{n}\beta_i$. For $m\geq 0$ and $1\leq\sigma\leq\infty$, define $L^{m,\sigma}(\Omega)=\{u=(u^\beta)_{|\beta|\leq m}\,|\,u^\beta\in L^\sigma(\Omega),\ |\beta|\leq m\}$. For u in $L^{m,\sigma}(\Omega)$, $\|u\|_{m,\sigma,\Omega}=\left(\sum_{|\beta|\leq m}\|u^\beta\|^\sigma_{L^\sigma(\Omega)}\right)^{\frac{1}{\sigma}}$ if $\sigma<\infty$, otherwise, denote $\|u\|_{m,\infty,\Omega}=\max_{|\beta|\leq m}\ \operatorname{ess\,sup}_{x\in\Omega}|u^\beta(x)|$. If w is an element in Sobolev space $W^{m,\sigma}(\Omega)$, let it correspond to the element $(D^\beta w)$ in $L^{m,\sigma}(\Omega)$, then $W^{m,\sigma}(\Omega)$ is a closed subspace of $L^{m,\sigma}(\Omega)$.

Let $\mathscr{K}$ be a set consisting of some n-simplexes or n-parallelotopes. Let K_h, for $h\in(0,1)$, be a finite subdivision of Ω. Assume that all the elements in K_h are in $\mathscr{K}$ and K_h satisfy the usual assumptions (see [2—5]).

Finite Element Spaces $U_{1,h}$. For each K in $\mathscr{K}$, we give two linear operators of interpolation, say $\Pi^0_K: C^1(K)\to P(K)$, $\Pi_{\partial K}: C^1(K)\to L^\infty(\partial K)$, and n finite dimensional spaces $N^{e_i}_K\subset P(K), 1\leq i\leq n$. Here $P(K)$ consists of all polynomials on K and $e_i=(\delta_{1i},\cdots,\delta_{ni})$. Then define $\Pi^{e_i}_K: C^1(K)\to N^{e_i}_K$, $1\leq i\leq n$, as follows, for any v in $C^1(K)$, $\Pi^{e_i}_K v$ is determined by the following equations,

$$1\leq i\leq n,\int_K p\Pi^{e_i}_K v\,dx=\int_{\partial K}p\Pi_{\partial K}vN_i\,ds-\int_K D^{e_i}p\Pi^0_K v\,dx,\quad\forall p\in N^{e_i}_K, \tag{1.1}$$

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where $N = (N_1, \cdots, N_n)^T$ is the unit outward normal of ∂K.

Define, for $h \in (0,1)$, $\Pi_h^1 : C^1(\overline{\Omega}) \to L^{1,\infty}(\Omega)$ by the way that for each u in $C^1(\overline{\Omega})$,

$$(\Pi_h^1 u)^\beta|_K = \Pi_K^\beta(u|_K), \quad K \in K_h, \quad |\beta| \leq 1. \tag{1.2}$$

The finite element spaces $U_{1,h}$, $\mathring{U}_{1,h}$ are obtained by setting

$$\begin{cases} U_{1,h} = \Pi_h^1 C^1(\overline{\Omega}), \\ \mathring{U}_{1,h} = \{ w \,|\, w = \Pi_h^1 u,\ u \in C^1(\overline{\Omega}) \text{ and } D^\beta u|_{\partial\Omega} = 0,\ |\beta| \leq 1 \}, \end{cases} \tag{1.3}$$

and they are used to approximate $W^{1,\sigma}(\Omega)$ and $\mathring{W}^{1,\sigma}(\Omega) \equiv \{ w \,|\, w \in W^{1,\sigma}(\Omega)$ and $w|_{\partial\Omega} = 0 \}$.

Suppose there exists a group of linearly independent functionals on $C^1(K)$, say $\varphi_{1,K}, \cdots, \varphi_{M,K}$, and polynomials $G_{i,K}$ and $g_{i,K} \in L^\infty(\partial K)$, $1 \leq i \leq M$, such that, for any v in $C^1(K)$,

$$\Pi_K^0 v = \sum_{i=1}^M \varphi_{i,K}(v) G_{i,K}, \Pi_{\partial K} v = \sum_{i=1}^M \varphi_{i,K}(v) g_{i,K}. \tag{1.4}$$

and $\varphi_{i,K}(v) = 0$ with $1 \leq i \leq M$ when $\Pi_K^0 v = 0$ and $\Pi_{\partial K} v = 0$. call $\varphi_{i,K}$ the parameters of $\Pi_K^0, \Pi_{\partial K}$.

For space N_K^β, $|\beta| = 1$, choose a basis of N_K^β, say $p_{j,K}^\beta$, $1 \leq j \leq \nu_\beta$. Denote the coordinate vector of $\Pi_K^\beta v$ with respect to this basis by $\zeta_{\beta,K}(v)$. Set $\zeta_K^1(v) = (\zeta_{e_1,K}^T(v), \cdots, \zeta_{e_n,K}^T(v))^T$ and $\Phi_K^1(v) = (\varphi_{1,K}(v), \cdots, \varphi_{M,K}(v))^T$, then equations (1.1) amounts to

$$A_K^1 \zeta_K^1(v) = Q_K^1 \Phi_K^1(v). \tag{1.5}$$

where A_K^1 is symmetric positive definite matrix and Q_K^1 is $(\sum_{i=1}^n L_{e_i}) \times M$ matrix.

Finite Element Spaces $U_{2,h}$. Now let $n = 2$. For each K in $\mathscr{K}$, we give four linear operators $\Pi_K^0 : C^2(K) \to P(K)$ and $\Pi_{\partial K}, \Pi_{\partial K}^S, \Pi_{\partial K}^N : C^2(K) \to L^\infty(\partial K)$ and five finite dimensional spaces N_K^β, $1 \leq |\beta| \leq 2$, consisting of polynomials. Then define $\Pi_K^\beta : C^2(K) \to N_K^\beta$, $0 < |\beta|$, as follows: for any v in $C^2(K)$, $\Pi_K^\beta v$ are determined by the following equations,

$$\begin{cases} i = 1, 2, \int_K p \Pi_K^{e_i} v \, dx = \int_{\partial K} p \Pi_{\partial K} v N_i \, ds - \int_K D^{e_i} p \Pi_K^0 v \, dx, \quad \forall p \in N_K^{e_i}, \\ \int_K p \Pi_K^{(2,0)} v \, dx = \int_{\partial K} p (N_1^2 \Pi_{\partial K}^N v - N_1 N_2 \Pi_{\partial K}^S v) ds - \int_K D^{e_1} p \Pi_K^{e_1} v \, dx, \quad \forall p \in N_K^{(2,0)}, \\ \int_K 2 p \Pi_K^{(1,1)} v \, dx = \int_{\partial K} p (2 N_1 N_2 \Pi_{\partial K}^N v + (N_1^2 - N_2^2) \Pi_{\partial K}^s v) ds - \int_K (D^{e_1} p \Pi_K^{e_2} v + \end{cases}$$

$$+ D^{e_2} p \Pi_K^{e_1} v) dx, \ \forall p \in N_K^{(1,1)}, \tag{1.6}$$

$$\int_K p \Pi_K^{(0,2)} v dx = \int_{\partial K} p (N_2^2 \Pi_{\partial K}^N v + N_1 N_2 \Pi_{\partial K}^s v) ds - \int_K D^{e_2} p \Pi_K^{e_2} v dx, \ \forall p \in N_K^{(0,2)}.$$

Define, for $h \in (0,1)$, $\Pi_h^2 : C^2(\overline{\Omega}) \to L^{2,\infty}(\Omega)$ as follows,

$$u \in C^2(\overline{\Omega}), \ (\Pi_h^2 u)^\beta|_K = \Pi_K^\beta (u|_K), \ |\beta| \le 2, K \in K_h. \tag{1.7}$$

Then finite element spaces $U_{2,h}$ and $\mathring{U}_{2,h}$, which are used to approximate $W^{2,\sigma}(\Omega)$ and $\mathring{W}^{2,\sigma}(\Omega)$, are obtained by setting,

$$\begin{cases} U_{2,h} = \Pi_h^2 C^2(\overline{\Omega}), \\ \mathring{U}_{2,h} = \{ w \mid w = \Pi_h^2 u, \ u \in C^2(\overline{\Omega}) \text{ with } D^\beta u|_{\partial\Omega} = 0 \text{ for all } |\beta| \le 2 \}. \end{cases} \tag{1.8}$$

Suppose there exists a group of linearly independent functionals $\varphi_{1,K}, \cdots, \varphi_{M,K}$ on $C^2(K)$, and $G_{j,K} \in P(K)$ and $g_{j,K}, g_{j,K}^s, g_{j,K}^N \in L^\infty(\partial K)$, $1 \le j \le M$, such that, for any v in $C^2(K)$,

$$\begin{cases} \Pi_K^0 v = \sum_{j=1}^M \varphi_{j,K}(v) G_{j,K}, & \Pi_{\partial K} v = \sum_{j=1}^M \varphi_{j,K}(v) g_{j,K}, \\ \Pi_{\partial K}^s v = \sum_{j=1}^M \varphi_{j,K}(v) g_{j,K}^s, & \Pi_{\partial K}^N v = \sum_{j=1}^M \varphi_{j,K}(v) g_{j,K}^{N}, \end{cases} \tag{1.9}$$

and $\varphi_{j,K}(v) = 0$ for $j = 1, \cdots, M$, when $\Pi_K^0 v = 0$ and $\Pi_{\partial K} v = \Pi_{\partial K}^s v = \Pi_{\partial K}^N v = 0$. $\varphi_{j,K}$ are called the parameters of Π_K^0, $\Pi_{\partial K}$, $\Pi_{\partial K}^s$ and $\Pi_{\partial K}^N$.

For space N_K^β, $0 < |\beta| \le 2$, we choose a basis of N_K^β, say $p_{j,K}^\beta$, $1 \le j \le L_\beta$. Denote the coordinate vector of $\Pi_K^\beta v$ with respect to this basis by $\zeta_{\beta,K}(v)$. Set $\zeta_K^2(v) = (\zeta_{(2,0),K}^T(v), \zeta_{(1,1),K}^T(v), \zeta_{(0,2),K}^T(v))^T$ and $\Phi_K^2(v) = (\varphi_{1,K}(v), \cdots, \varphi_{M,K}(v))^T$. Then the last three equations of (1.6) amount to

$$A_K^2 \zeta_K^2(v) = Q_K^2 \Phi_K^2(v), \tag{1.10}$$

where A_K^2 is a symmetric positive definite matrix and Q_K^2 is $\left(\sum_{|\beta|=2} L_\beta\right) \times M$ matrix.

Finnally, we must point out that the conforming element spaces, nonconforming element spaces and quasi-conforming element spaces are special cases of $U_{1,h}$ and $U_{2,h}$ (see [2]).

2. Some Properties of Finite Element Spaces

First, we give some definitions. Choose a fixed n-simplex (or n parallelotope) $\hat{K}$. Then for each n-simplex (or n-parallelotope) K, there exists an affine transformation, $F_K \hat{x} = B_K \hat{x} + b_K$, such that $K = F_K \hat{K}$, with B_K is an $n \times n$

nonsingular matrix and b_K a vector in R^n.

We define, for every function w defined on K (or on ∂K),

$$\widehat{w}(\widehat{x}) = w(F_K \widehat{x}), \quad \forall \widehat{x} \in \widehat{K} \ (\text{or } \partial\widehat{K}). \tag{2.1}$$

And for the linear functional φ on $C^r(K)$, we define a linear functional $\widehat{\varphi}$ on $C^r(\widehat{K})$, such that,

$$\widehat{\varphi}(\widehat{w}) = \varphi(w), \quad \forall \widehat{w} \in C^r(\widehat{K}). \tag{2.2}$$

where w and $\widehat{w}$ satisfy (2.1).

$\{\Pi_K^0, \Pi_{\partial K}, N_K^\beta\}$ is called a properly affine continuous family, if i) for a sequence of n-simplexes or n-parallelotopes $\{K_t\}_{t=0}^{\infty}$, if $K_t \in \mathscr{K}$, $t \in \mathbf{N}$, and B_{K_t} converges to B_{K_0}, then $K_0 \in \mathscr{K}$, and for $1 \le j \le M$, $\widehat{\varphi}_{j,K_t} \to \widehat{\varphi}_{j,K_0}$ in the dual space of $C^1(\widehat{K})$, $\widehat{G}_{j,K_t}$ converges to $\widehat{G}_{j,K_0}$ uniformly and $\widehat{g}_{j,K_t}$ converges to $\widehat{g}_{j,K_0}$ in $L^\infty(\partial\widehat{K})$, and for $0 < |\beta| \le 1$, $1 \le i \le L_\beta$, $\widehat{p}_{i,K_t}^{\beta}$ converges to $\widehat{p}_{i,K_t}^{\beta}$ uniformly; and ii) for each $K \in \mathscr{K}$ and constant $C > 0$, denote $\widetilde{K} = \{\widetilde{x} \mid \widetilde{x} = Cx, \forall x \in K\}$, and for function w defined on K or ∂K, define $\widetilde{w}(\widetilde{x}) = w(C^{-1}\widetilde{x})$, $\widetilde{x} \in \widetilde{K}$ or $\partial\widetilde{K}$, then $\widetilde{K} \in \mathscr{K}$, and $\widetilde{\Pi_K^0 w} = \Pi_{\widetilde{K}}^0 \widetilde{w}$ and $\widetilde{\Pi_{\partial K} w} = \Pi_{\partial\widetilde{K}} \widetilde{w}$ for $w \in C^1(K)$, and for $0 < |\beta| \le 1$, $\widetilde{p}_{i,K}^{\beta}$, $1 \le i \le L_\beta$, is a basis of $N_{\widetilde{K}}^\beta$.

The properly affine continuous family is the generalization of the affine family [6]. The operators Π_K^0, $\Pi_{\partial K}$ actually used all satisfy the conditions of the properly affine continuous family (see [3]). And the simplest case of N_K^β satisfying the conditions i) and ii) is that $N_K^\beta = \{p \mid p(x) = \widehat{p}(F_K^{-1}x), \forall \widehat{p} \in N_{\widehat{K}}^\beta\}$.

$\{\Pi_K^0, \Pi_{\partial K}, N_K^\beta\}$ is strongly continuous if for $K \in \mathscr{K}$ and every $(n-1)$-dimensional surface F of K, $\Pi_{\partial K} w|_F \in C(F)$ when $w \in C^1(K)$, and there exists a lineear continuous functional q_F on $C(F)$ such that, i) $\{q_F\}$ is an affine family, i.e., for $w \in C(F)$, $q_F(w) = q_{\widehat{F}}(\widehat{w})$ when $F = F_K \widehat{F}$, ii) $\widehat{q_{\widehat{F}}}(1) \neq 0$, and iii) if F is an $(n-1)$-dimensional common surface of K' and N'', then $q_F(\Pi_{\partial K'} w|_F) = q_F(\Pi_{\partial K''} w|_F)$ when $w \in C^1(K' \cup K'')$.

The definitions of the properly affine continuous family and the strong continuity for $\{\Pi_K^0, \Pi_{\partial K}, \Pi_{\partial K}^s, \Pi_{\partial K}^N, N_K^\beta\}$ can be given in a similar way. For the details see [3]. From [3] we know that $\{\Pi_K^0, \Pi_{\partial K}, N_K^\beta\}$ and $\{\Pi_K^0, \Pi_{\partial K}^s, \Pi_{\partial K}^N, N_K^\beta\}$ actually used are all strongly continuous.

$\{\Pi_K^0, \Pi_{\partial K}, N_K^\beta\}$ passes the IPT test, if for $K \in \mathscr{K}$ there exists another polynomial interpolation operator $\widetilde{\Pi}_{\partial K}: C^1(K) \to L^\infty(\partial K)$ with the properties: i) $\{\Pi_K^0, \widetilde{\Pi}_{\partial K}, N_K^\beta\}$ is a properly affine continuous family and the parameters of Π_K^0, $\widetilde{\Pi}_{\partial K}$ are the linear combinations of those of Π_K^0, $\Pi_{\partial K}$; ii) $\widetilde{\Pi}_{\partial K} p = p|_{\partial K}$ for

$p\in P_0(K)$; iii) if K and $K'\in\mathscr{X}$ and $F=K\cap K'$ is $(n-1)$ dimensional, then $\int_F\widetilde{\Pi}_{\partial K}w\,ds=\int_F\widetilde{\Pi}_{\partial K'}w\,ds$ for $\forall w\in C^1(K\cup K')$; and iv) for $\forall w\in C^1(K)$ and $K\in\mathscr{X}$, $\int_{\partial K}N_i(\Pi_{\partial K}-\widetilde{\Pi}_{\partial K})w\,ds=0$, $1\le i\le n$. Where $P_t(K)$ is the space consisting of all polynomials with degree not greater than t.

Similarly, $\{\Pi^0_K,\Pi_{\partial K},\Pi^s_{\partial K},\Pi^N_{\partial K},N^\beta_K\}$ passes the IPT test, if for $K\in\mathscr{X}$, there exist another two linear operators $\widetilde{\Pi}^s_{\partial K},\widetilde{\Pi}^N_{\partial K}$: $C^2(K)\to L^\infty(\partial K)$ with the properties: i) $\{\Pi^0_K,\Pi_{\partial K},\widetilde{\Pi}^s_{\partial K},\widetilde{\Pi}^N_{\partial K},N^\beta_K\}$ is a properly affine continuous family and the parameters of $\Pi^0_K,\Pi_{\partial K},\widetilde{\Pi}^s_{\partial K},\widetilde{\Pi}^N_{\partial K}$ are the linear combinations of those of Π^0_K, $\Pi_{\partial K},\Pi^s_{\partial K}$; ii) $\widetilde{\Pi}^s_{\partial K}p=\frac{\partial p}{\partial s}\big|_{\partial K}$ and $\widetilde{\Pi}^N_{\partial K}p=\frac{\partial p}{\partial N}\big|_{\partial K}$ for $p\in P_1(K)$; iii) if K and $K'\in\mathscr{X}$ and F is a common side of both K and K', then $\int_F(\widetilde{\Pi}^N_{\partial K}w+\widetilde{\Pi}^N_{\partial K'}w)ds=\int_F(\widetilde{\Pi}^s_{\partial K}w-\Pi^s_{\partial K'}w)ds=0$ for $\forall w\in C^2(K\cup K')$; and iv) for any $w\in C^2(K)$,

$$\int_{\partial K}\begin{bmatrix}N_1^2(\Pi^N_{\partial K}-\widetilde{\Pi}^N_{\partial K})w-N_1N_2(\Pi^s_{\partial K}-\widetilde{\Pi}^s_{\partial K})w\\ 2N_1N_2(\Pi^N_{\partial K}-\widetilde{\Pi}^N_{\partial K})w+(N_1^2-N_2^2)(\Pi^s_{\partial K}-\widetilde{\Pi}^s_{\partial K})w\\ N_2^2(\Pi^N_{\partial N}-\widetilde{\Pi}^N_{\partial K})w+N_1N_2(\Pi^s_{\partial K}-\widetilde{\Pi}^s_{\partial K})w\end{bmatrix}ds=0.$$

$\{\Pi^0_K,\Pi_{\partial K},N^\beta_K\}$ satisfies the rank condition of element if the rank of matrix Q^1_K is $M-1$ for $\forall K\in\mathscr{X}$. $\{\Pi^0_K,\Pi_{\partial K},\Pi^s_{\partial K},\Pi^N_{\partial K},N^\beta_K\}$ satisfies the rank condition of element if the rank of matrix Q^2_K is $M-3$ for $\forall K\in\mathscr{X}$.

By the way used in [3,4], we can get some properties of $U_{1,h}$ and $U_{2,h}$. We list them without proof.

Theorem I Let $1<\sigma<\infty$ and the following (H1) and (H2) be true:
(H1). $\{\Pi^0_K,\Pi_{\partial K},N^\beta_K\}$ is a properly affine continuous family, satisfies the rank condition of element, has the strong continuity and passes the IPT test.
(H2). There exist two integers r_1 and r_2 such that, $r_i\ge 1$ for $i=1,2$, Π^0_K can be extended to a bounded linear operator from $W^{r_1+1,\sigma}(K)$ to $W^{1,\infty}(K)$ and $\Pi_{\partial K}$ to a bounded linear operator from $W^{r_2+1,\sigma}(K)$ to $L^\infty(\partial K)$, and $\Pi^0_Kp=p$ for $p\in P_{r_1}(K)$, $\Pi_{\partial K}p=p|_{\partial K}$ for $p\in P_{r_2}(K)$; and for $i=1,\cdots,n$, $P_0(K)\subset N^{e_i}_K$.
Then then following conclusions are true: I) if $\mu\ge\sigma$ and $\mu\le n\sigma/(n-\sigma)$ when $n>\sigma$, $\mu<\infty$ when $n=\sigma$ and $\mu\le\infty$ when $n<\sigma$, then there is a constant C independent of h, such that,

$$u_h\in U_{1,h},\ \|u^0_h\|_{L^\mu(\Omega)}\le C\|u_h\|_{1,\sigma,\Omega},\tag{2.3}$$

is true for h sufficiently small; (II) if $\mu\ge\sigma$ and $\mu<n\sigma/(n-\sigma)$ when $n>\sigma$,

$\mu<\infty$ when $n=\sigma$ and $\mu\leq\infty$ when $n<\sigma$, then for $\forall u$ in $W^{1,\sigma}(\Omega)$, $\lim\limits_{h\to 0}\inf\limits_{v\in U_{1,h}}$ $\{\|u-v^0\|_{L^\mu(\Omega)}+\|u-v\|_{1,\sigma,\Omega}\}=0$, and for $\forall u$ in $\mathring{W}^{1,\sigma}(\Omega)$, $\lim\limits_{h\to 0}\inf\limits_{v\in\mathring{U}_{1,h}}\{\|u-v^0\|_{L^\mu(\Omega)}$ $+\|u-v\|_{1,\sigma,\Omega}\}=0$; III) let μ satisfy the requirement in II), then for each bounded sequence $\{u_m\}$ of $L^{1,\sigma}(\Omega)$ with $u_m\in U_{1,h_m}$ for $m\in\mathbf{N}$ and $h_m\to 0$ as $m\to\infty$, there exists a subsequence $\mathbf{N}'$ of $\mathbf{N}$ and u_0 in $W^{1,\sigma}(\Omega)$, such that, $\{u_m\}_{m\in\mathbf{N}'}$ weakly converges, in $L^{1,\sigma}(\Omega)$ sense, to u_0 and $\lim\limits_{m\in\mathbf{N}',m\to\infty}\|u_0-u_m^0\|_{L^\mu(\Omega)}=0$; if $U_{1,h}$ and $W^{1,\sigma}(\Omega)$ are replaced by $\mathring{U}_{1,h}$ and $\mathring{W}^{1,\sigma}(\Omega)$ respectively, the conclusion is also true; IV) there is a constant C independent of h, such that,

$$u_h\in\mathring{U}_{1,h},\ \|u_h\|_{1,\sigma,\Omega}\leq C\sum_{|\beta|=1}\|u_h^\beta\|_{L^\sigma(\Omega)} \tag{2.4}$$

hold for h sufficiently small.

Theorem 2 Let $1<\sigma<\infty$ and the following (H3) and (H4) hold:

(H3). There exist integers $r_i\geq 2(1\leq i\leq 4)$, such that, Π_K^0 can be extended to a bounded linear operator from $W^{r_1+1,\sigma}(K)$ to $W^{2,\omega}(K)$, and $\Pi_{\partial K}$, $\Pi_{\partial K}^s$ and $\Pi_{\partial K}^N$ to bounded linear operators from $W^{r_2-1,\sigma}(K)$, $W^{r_3-1,\sigma}(K)$ and $W^{r_4-1,\sigma}(K)$ to $L(\partial K)$ respectively, and $\Pi_K^0 p=p$ for $p\in P_{r_1}(K)$, $\Pi_{\partial K}p=p|_{\partial K}$ for $p\in P_{r_2}(K)$, $\Pi_{\partial K}^s p=\frac{\partial p}{\partial s}|_{\partial K}$ for $p\in P_{r_3}(K)$ and $\Pi_{\partial K}^N p=\frac{\partial p}{\partial N}|_{\partial K}$ for $p\in P_{r_4}(K)$; and for $0<|\beta|\leq 2$, $P_{2-|\beta|}(K)\subset N_K^\beta$.

(H4). $\{\Pi_K^0, \Pi_{\partial K}, \Pi_{\partial K}^s, \Pi_{\partial K}^N, N_K^\beta\}$ is a properly affine continuous family, satisfies the rank condition of element, has the strong continuity and passes the IPT test. Then the following conclusions are true: I) if for $j=0,1$, $\mu_j\geq\sigma$ and $\mu_j\leq 2\sigma/(2-2\sigma+j\sigma)$ when $2>(2-j)\sigma$, $\mu_j<\infty$ when $2=(2-j)\sigma$ and $\mu_j\leq\infty$ when $2<(2-j)\sigma$, then there exists a constant C independent of h, such that,

$$u_h\in U_{2,h},\ \sum_{j=0,1}\sum_{|\beta|=j}\|u_h^\beta\|_{L^{\mu_j}(\Omega)}\leq C\|u_h\|_{2,\sigma,\Omega} \tag{2.5}$$

hold for h sufficiently small; II) if for $j=0,1$, $\mu_j\geq\sigma$ and $\mu_j<2\sigma/(2-2\sigma+j\sigma)$ when $2>(2-j)\sigma$, $\mu_j<\infty$ when $2=(2-j)\sigma$ and $\mu_j\leq\infty$ when $2<(2-j)\sigma$, then for $\forall u\in W^{2,\sigma}(\Omega)$,

$$\lim_{h\to 0}\inf_{v\in U_{2,h}}\{\|u-v\|_{2,\sigma,\Omega}+\sum_{j=0,1}\sum_{|\beta|=j}\|D^\beta u-v^\beta\|_{L^{\mu_j}(\Omega)}\}=0;$$

if $u\in\mathring{W}^{2,\sigma}(\Omega)$, the $U_{2,h}$ in above equality can be replaced by $\mathring{U}_{2,h}$; III) let μ_0, μ_1 satisfy the assumption in II), then for every bounded sequence $\{u_m\}$ in $L^{2,\sigma}(\Omega)$ with $u_m\in U_{2,h_m}$ for $m\in\mathbf{N}$ and $h_m\to 0$ as $m\to\infty$, there is a subsequence $\mathbf{N}'$ of $\mathbf{N}$ and u_0 in $W^{2,\sigma}(\Omega)$, such that $\{u_m\}_{m\in\mathbf{N}'}$ weakly converges, in $L^{2,\sigma}(\Omega)$ sense, to u_0 and

$$\lim_{m\to\infty,\, m\in \mathbf{N}} \sum_{j=0,1} \sum_{|\beta|=j} \|D^\beta u_0 - u_m^\beta\|_{L^\mu(\Omega)} = 0;$$

if $U_{2,h}$ and $W^{2,\sigma}(\Omega)$ are replaced by $\overset{\circ}{U}_{2,h}$ and $\overset{\circ}{W}{}^{2,\sigma}(\Omega)$ respectively, the conclusion is also true; Ⅳ) there is a constant C independent of h, such that,

$$u_h \in \overset{\circ}{U}_{2,h},\ \|u_h\|_{2,\sigma,\Omega} \leq C \sum_{|\beta|=2} \|u^\beta\|_{L^\sigma(\Omega)}, \tag{2.6}$$

hold for h sufficiently small.

It can be shown that the spaces $U_{1,h}$, constructed by well-known conforming elemints, Wilson's element and the element of Crouzeit-Raviart, satisfy (H1) and (H2)(see [8]), and that the spaces $U_{2,h}$, constructed by the element of Fraeijs de Veubake[8], Morley's element and 9-parameter, 12-parameter and 15-parameter quasi-conforming elements, satisfy (H3) and (H4)(see [8] and [2,3]).

3. Navier-Stokes' Equations

This section is devoted to the finite element method for n-dimensional stationary Navier-Stokes equations with $n \leq 3$. Let $f \in (L^2(\Omega))^n$ and $\lambda > 0$ be Reynolds number, the problem is finding $(u,p) \in (\overset{\circ}{W}{}^{1,2}(\Omega))^n \times L^2(\Omega)$, such that,

$$-\lambda^{-1}\triangle u + (u\cdot\nabla)u + \nabla p = f,\quad \nabla\cdot u = 0 \text{ in } \Omega. \tag{3.1}$$

Set $X = (L^{1,2}(\Omega))^n$, $U_0 = (\overset{\circ}{W}{}^{1,2}(\Omega))^n$, $U_h = (\overset{\circ}{U}_{1,h})^n$, denote $\tilde{X} = \{v \mid v = \sum_{i=1}^{L} c_i v_i, v_i \in \bigcup_{h\in(0,1)} U_h,\ c_i \in R,\ L \in \mathbf{N}\}$. Define $A: X \to X$, $G: \tilde{X} \to X$ as follows,

$$(Au, v) = \sum_{i,j=1}^{n} \int_\Omega u_i^{e_j} v_i^{e_j} dx,\quad \forall u, v \in X, \tag{3.2}$$

$$(Gu, v) = -\sum_{i,j=1}^{n} \int_\Omega u_i^0 u_j^0 v_i^{e_j} dx - \sum_{i=1}^{n} \int_\Omega f_i v_i^0 dx,\quad \forall u \in \tilde{X}, v \in X, \tag{3.3}$$

where $(\cdot,\cdot)$ is the product of X. For $v \in X$ and $p \in L^2(\Omega)$, set

$$b(p, v) = \sum_{i=1}^{n} \int_\Omega p v_i^{e_i} dx. \tag{3.4}$$

Let $L_0^2(\Omega) = \{p \mid p \in L^2(\Omega), \int_\Omega p dx = 0\}$. Consider the problem:

$$\begin{gathered}(u,p) \in U_0 \times L_0^2(\Omega),\ (Au, v) + \lambda((Gu, v) - b(p, v) - b(q, u)) = 0, \\ \forall (v, q) \in U_0 \times L_0^2(\Omega),\end{gathered} \tag{3.5}$$

It is well known that the solution of (3.1) is the solution of (3.5), and equation (3.5) always has a solution for $\lambda > 0$.

For $h\in(0,1)$, choose a finite dimensional subspace W_h of $L_0^2(\Omega)$. The finite element method for problem (3.5) is solving the following problem:

$$(u_h,p_h)\in U_h\times W_h,\ (Au_h,v_h)+\lambda((Gu_h,v_h)-b(p_h,v_h)-b(q_h,u_h))=0,\quad \forall(v_h,q_h)\in U_h\times W_h. \tag{3.6}$$

Denote $X_0=\{v\,|v\in U_0,\ b(q,v)=0,\ \forall q\in L^2(\Omega)\}$, $X_h=\{v\,|v\in U_h,\ b(q_h,v)=0,\ \forall q_h\in W_h\}$. Then equations (3.5) and (3.6) are equivalent to the following (3.7) and (3.8) respectively,

$$\begin{cases} u\in X_0,\ (Au_0,v)+\lambda(Gu,v)=0,\ \forall v\in X_0,\\ p\in L_0^2(\Omega),\ b(p,v)=\lambda^{-1}(Au,v)+(Gu,v),\ \forall v\in U_0, \end{cases} \tag{3.7}$$

$$\begin{cases} u_h\in X_h,\ (Au_h,v_h)+\lambda(Gu_h,v_h)=0,\ \forall v_h\in X_h,\\ p_h\in W_h,\ b(p_h,v_h)=\lambda^{-1}(Au_h,v_h)+(Gu_h,v_h),\ \forall v_h\in U_h. \end{cases} \tag{3.8}$$

Lemma 1 Let (H1) and (H2) hold for $\sigma=2$. Then there exists $h_0\in(0,1)$, such that, i) $G:\tilde{X}\to X$ is infinitely Gateaux differentiable, $d^rG\in L_r(X,X)$, $1\leq r<\infty$, and for h, h' in $[0,h_0)$, $T_hG:X_{h'}\to X$ is infinitely Fréchet differentiable; ii) the set $\{\|T_hd^rG(v)\|_{L_r(X_h,X)}|\,v\in X_{h'}\cap B,\ h,h'\in[0,h_0),\ 0\leq r<\infty\}$ is bounded when B is a bounded set in X; iii) if $v_m\in X_{h_m}$, $m\in\mathbf{N}$, and v_m converges to v_0 and $h_m\to0$, then for $0\leq r<\infty$, $\lim\|T_{h_m}(d^rG(v_m)-d^rG(v_0))\|_{L_r(X_{h_m},X)}=0$; iv) if $v_m\in X_{h_m}$, $m\in\mathbf{N}$, and v_m weakly converges to 0 and $h_m\to0$, then for $\forall u\in X_0$, $\lim(dG(u)v_m,v_m)=0$; v) A is uniformly U_h-elliptic.

It is not difficult to show lemma 1 by theorem 1 and Sobolev's embedding theorem.

Lemma 2 Let (H1) and (H2) hold with $\sigma=2$. Suppose that there is a constant C independent of h, such that, for $h\in(0,h_0)$,

$$\inf_{q_h\in W_h}\sup_{v_h\in U_h}|b(q_h,v_h)|/(\|v_h\|\|q_h\|_{L^2(\Omega)})\geq C>0, \tag{3.9}$$

$$q\in L_0^2(\Omega),\quad \lim_{h\to0}\inf_{q_h\in W_h}\|q-q_h\|_{L^2(\Omega)}=0. \tag{3.10}$$

Then i) for u_h satisfying the first equation of (3.8), the second one of (3.8) has a unique solution p_h; ii) $\{X_h\}$, X_0 has the approximabilty and the weak closeness.

Proof Conclusion i) is obvious when (3.9) is true. If $v_m\in X_{h_m}$ for $m\in\mathbf{N}$, and v_m weakly converges to v_0 and $h_m\to0$, theorem 1 tells us $v_0\in U_0$. For $\forall q\in L_0^2(\Omega)$, there exist $q_m\in W_{h_m}$ such that $\lim\|q-q_m\|_{L^2(\Omega)}=0$. Thus $b(q,v_0)=\lim b(q_m,v_m)=0$, that is, $v_0\in X_0$. The weak closeness is true.

If $u_0\in X_0$, let u_h be the solution of the equations:

$$(u_h,p_h)\in U_h\times W_h,\ (Au_h,v_h)-b(p_h,v_h)-b(q_h,u_h)=(Au_0,v_h),\quad \forall(v_h,q_h)\in U_h\times W_h,$$

then $u_h \in X_h$ and $\lim \|u_0 - u_h\| = 0$ by theorem 1, (3.9) and (3.10) and Brezzi's theorem. The approximability is true.

Now we can apply the results in [1] to prove the convergence of the solutions of equation (3.6).

Theorem 3 Let (H1) and (H2) hold for $\sigma = 2$, and (3.9) and (3.10) be true. Then i) if Λ is a bounded closed interval and $\{(\lambda, u(\lambda)) \mid \lambda \in \Lambda\}$ is a branch of nonsingular solutions of the first equation of (3.7), then there exists, for h sufficiently small, a unique branch of solutions $\{(u_h(\lambda), p_h(\lambda)) \mid \lambda \in \Lambda\}$ of equation (3.6), such that, for $i \geq 0$,

$$\lim_{h\to 0} \sup_{\lambda \in \Lambda} \{ \|d^i (u(\lambda) - u_h(\lambda))\| + \|d^i (p(\lambda) - p_h(\lambda))\|_{L^2(\Omega)} \} = 0,$$

where $(u(\lambda), p(\lambda))$ is a solution of (3.5) for $\lambda \in \Lambda$; ii) if $(\lambda_0, u_0) \in R \times X_0$ is a limit point of the first equation of (3.7), then there is $a_0 > 0$ and a branch of solutions $\{(\lambda(a), u(a), p(a)) \mid |a| \leq a_0\}$ of equation (3.5) with $(\lambda(0), u(0)) = (\lambda_0, u_0)$, and for h sufficiently small, equation (3.6) has a unique branch of solutions $\{(\lambda_h(a), u_h(a), p_h(a)) \mid |a| \leq a_0\}$, such that, for $i \geq 0$,

$$\lim_{h\to 0} \sup_{|a| \leq a_0} \{ \|d^i (\lambda(a) - \lambda_h(a))| + \|d^i (u(a) - u_h(a))\| + \|d^i (p(a) - p_h(a))\|_{L^2(\Omega)} \} = 0.$$

It can be verified that the conforming elements and nonconforming elements, discussed in [11] for the Stokes problems, satisfy the condition of theorem 3. And it is valid to use them to solve equation (3.5).

4. Navier-Stokes' Equations in the Stream Function Formulation

The finite element method for equation (3.5) requires the condition (3.9), and it is very difficuilt to verify it. In the case of $n = 2$, we can introduce the stream function and get a equation of 4th order,

$$\begin{cases} \lambda^{-1}\Delta^2\varphi + \dfrac{\partial}{\partial x_1}\left(\varphi \dfrac{\partial}{\partial x_2}\Delta\varphi\right) - \dfrac{\partial}{\partial x_2}\left(\varphi \dfrac{\partial}{\partial x_1}\Delta\varphi\right) = f, \quad \text{in } \Omega \\ \varphi|_{\partial\Omega} = \dfrac{\partial\varphi}{\partial N}\Big|_{\partial\Omega} = 0, \end{cases} \tag{4.1}$$

where $f \in L^2(\Omega)$ and p disappears.

Let $X = L^{2,2}(\Omega)$, $X_0 = \mathring{W}^{2,2}(\Omega)$, $X_h = \mathring{U}_{2,h}$, $h \in (0, 1)$, Set $\tilde{X} = \{\psi \mid \psi = \sum_{i=1}^{L} c_i \psi_i$, $c_i \in R$, $L \in \mathbf{N}$, $\psi_i \in \bigcup_{h \in [0,1)} X_h\}$. Define A: $X \to X$, G: $R \times \tilde{X} \to X$ as follows,

$$(A\varphi, \psi) = \int_\Omega (\varphi^{(2,0)}\psi^{(2,0)} + 2\varphi^{(1,1)}\psi^{(1,1)} + \varphi^{(0,2)}\psi^{(0,2)})dx, \quad \forall \varphi, \psi \in X, \tag{4.2}$$

$$(G(\lambda, \varphi), \psi) = \lambda \int_\Omega \{ (\varphi^{(1,0)}\varphi^{(1,0)} - \varphi^{(0,1)}\varphi^{(0,1)})\psi^{(1,1)} +$$

$$+ \varphi^{(1,0)}\varphi^{(0,1)}(\psi^{(0,2)} - \psi^{(2,0)}) - f\psi^{(0,0)}\} dx, \quad \forall \varphi \in \tilde{X}, \psi \in X, \tag{4.3}$$

The weak form of problem (4.1) is

$$\varphi \in X_0, \quad (A\varphi, \psi) + (G(\lambda, \varphi), \psi) = 0, \quad \forall \psi \in X_0. \tag{4.4}$$

For $\lambda > 0$, (4.4) has at least one solution. The finite element approximation of problem (4.4) is the following problem,

$$\varphi_h \in X_h, \quad (A\varphi_h, \psi_h) + (G(\lambda, \varphi_h), \psi_h) = 0, \quad \forall \psi_h \in X_h. \tag{4.5}$$

By the way used in the above section and theorem 2, we can prove the following results.

Theorem 4 Let (H3) and (H4) hold for $\sigma = 2$. Then i) if Λ is a bounded closed interval and $\{(\lambda, \varphi(\lambda)) | \lambda \in \Lambda\}$ is a branch of nonsingular solutions of equation (4.4), then there exists, for h sufficiently small, a unique branch of solutions $\{(\lambda, \varphi_h(\lambda)) | \lambda \in \Lambda\}$ of equation (4.5), such that, for $i \geq 0$, $\lim\limits_{h \to 0} \sup\limits_{\lambda \in \Lambda} \| d^i (\varphi(\lambda) - \varphi_h(\lambda)) \| = 0$; ii) if (λ_0, φ_0) is a limit point of equation (4.4), then there exist $a_0 > 0$ and a branch of solutions $\{(\lambda(a), \varphi(a)) | |a| \leq a_0\}$ of equation (4.4) with $(\lambda(0), \varphi(0)) = (\lambda_0, \varphi_0)$, and equation (4.5) has, for h sufficiently small, a unique branch of solutions $\{(\lambda_h(a), \varphi_h(a)) | |a| \leq a_0\}$ such that $\lim\limits_{h \to 0} \sup\limits_{|a| \leq a_0} \{ |d^i(\lambda(a) - \lambda_h(a))| + \| d^i(\varphi(a) - \varphi_h(a)) \| \} = 0$ for $i \geq 0$.

From the above theorem, we can conclude that the well-known conforming elements, the nonconforming elements, such as Adini's element and Morley's element etc., and 9-parameter, 12-parameter and 15-parameter quasi-conforming elements are convergent for equation (4.4) in the cases of nonsingular and limit points.

5. The von Karman Equations

In this section, set $n = 2$ and consider the finite element method for the von Karman equation,

$$\begin{cases} \triangle^2 \varphi_1 = -[\varphi_2, \varphi_2]/2, & \text{in}\Omega \\ \triangle^2 \varphi_2 = [\varphi_1, \varphi_2] + \lambda_1 [\bar{\varphi}, \varphi_2] + \lambda_2 f, & \text{in}\Omega \\ \varphi_1 |_{\partial\Omega} = \varphi_2 |_{\partial\Omega} = \dfrac{\partial \varphi_1}{\partial N} |_{\partial\Omega} = \dfrac{\partial \varphi_2}{\partial N} |_{\partial\Omega} = 0, \end{cases} \tag{5.1}$$

where φ_1 is the Airy stress function, φ_2 is the vertical deflection of the plate, $\lambda_2 f$ is an external vertical load on the plate, $f \in L^2(\Omega)$ and $\bar{\varphi} \in W^{3,2}(\Omega)$ are known functions, and for $u, v \in W^{2,2}(\Omega)$,

$$[u,v]=\frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_1^2}-2\frac{\partial^2 u}{\partial x_1\partial x_2}\frac{\partial^2 v}{\partial x_1\partial x_2}. \tag{5.2}$$

Let $\mathbf{X}=(\mathbf{L}^{2,2}(\Omega))^2$, $\mathbf{X}_0=(\mathbf{W}^{2,2}(\Omega))^2$, $\mathbf{X}_h=(\mathbf{U}_{2,h})^2$ for $h\in(0,1)$, $\tilde{\mathbf{X}}=\{\psi\,|\psi=\sum_{i=1}^{L}c_i\psi_i,\ \psi_i\in\bigcup_{h\in(0,1)}\mathbf{X}_h,\ L\in\mathbf{N}\}$. Define A: $\mathbf{X}\to\mathbf{X}$, G: $\mathbf{R}^2\times\tilde{\mathbf{X}}\to\mathbf{X}$ as follows,

$$(A\varphi,\psi)=\sum_{i=1}^{2}\int_\Omega(\varphi_i^{(2,0)}\psi_i^{(2,0)}+2\varphi_i^{(1,1)}\varphi_i^{(1,1)}+\varphi_i^{(0,2)}\psi_i^{(0,2)})\mathrm{d}x,\ \forall\varphi,\psi\in\mathbf{X}, \tag{5.3}$$

$$(G(\lambda,\varphi),\psi)=(\varphi_2,\varphi_2,\psi_1)/2-B(\varphi_1,\varphi_2,\psi_2)-\lambda_1B(\bar{\varphi},\varphi_2,\psi_2)-\lambda_2\int_\Omega f\psi_2^{(0,0)}\mathrm{d}x,$$

$$\forall\varphi\in\tilde{\mathbf{X}},\quad\psi\in\mathbf{X}, \tag{5.3}$$

where, for u,v and w in $\mathbf{L}^{2,2}(\Omega)$,

$$B(u,v,w)=\int_\Omega\{(u^{(1,0)}v^{(0,1)}+u^{(0,1)}v^{(1,0)})w^{(1,1)}-$$

$$-u^{(1,0)}v^{(1,0)}w^{(2,0)}-u^{(0,1)}v^{(0,1)}w^{(0,2)}\}\mathrm{d}x. \tag{5.5}$$

if the right hand of (5.5) makes sense.

The weak form of (5.1) is the problem,

$$\varphi\in\mathbf{X}_0,\quad(A\varphi,\psi)+(G(\lambda,\varphi),\psi)=0,\quad\forall\psi\in\mathbf{X}_0. \tag{5.6}$$

And the finite element method for (5.6) is the following problem,

$$\varphi_h\in\mathbf{X}_h,\quad(A\varphi_h,\psi_h)+(G(\lambda,\varphi_h),\psi_h)=0,\quad\forall\psi_h\in\mathbf{X}_h \tag{5.7}$$

For the solutions of (5.7), we have the similar conclusions.

Theorem 5 Let (H3) and (H4) hold for $\sigma=2$. Then i) if Λ is a bounded closed set in $\mathbf{R}^2$ and $\{(\lambda,\psi(\lambda))\,|\lambda\in\Lambda\}$ is a branch of nonsingular solutions of equation (5.6), then equation (5.7) has, for h sufficiently small, a unique branch of solution $\{(\lambda,\varphi_h(\lambda))\,|\lambda\in\Lambda\}$, such that for $i\geq0$,

$\lim_{h\to0}\sup_{\lambda\in\Lambda}\|\mathrm{d}^i(\varphi(\lambda)-\varphi_h(\lambda))\|_{L_i(\mathbf{R}^2,\mathbf{X})}=0$; ii) if (λ^0,φ^0) is a limit point of equation (5.6), then there is $a_0>0$ and a branch of solution $\{(\lambda(t),\varphi(t))\,|\,t\in S_2^{a_0}\}$ with $(\lambda(0),\varphi(0))=(\lambda^0,\varphi^0)$, and for h sufficiently small, equation (5.7) has a unique branch of solution $\{(\lambda_h(t),\varphi_h(t))\,|t\in S_2^{a_0}\}\}$ such that for $i\geq0$,

$\lim_{h\to0}\sup_{t\in S_2^{a_0}}\{\|\mathrm{d}^i(\lambda(t)-\lambda_h(t))\|-\|\mathrm{d}^i(\varphi(t)-\varphi_h(t))\|_{L_i(\mathbf{R}^2,\mathbf{X})}\}=0$; iii) now set the second component of λ be one, thus equations (5.6) and (5.7) are equations only containing one parameter λ_1, if (λ_1^0,φ^0) is a simple bifurcation point of (5.6), then there exists a neighborhood U of (λ_1^0,φ^0) in $\mathbf{R}\times\mathbf{X}$ and $h_0\in(0,1)$, such that, for $h\leq h_0$, the set $\mathscr{S}_h$ of the solutions of (5.7) contained in U consists of two infinitely differentiable branches and the distance between $\mathscr{S}_h$ and the set of the solutions of (5.6) contained in U converges to 0 as $h \to 0$.

From theorem 5, we can conclude that the well-known conforming elements, the nonconforming elements such as Adini's element and Morley's element, and 9-parameter, 12-parameter and 15-parameter quasi-conforming elements are valid for equation (5.6).

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