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Nearly Zero Boolean Idempotent Matrices*

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Abstract We establish a characterization theorem for a nearly zero Boolean idempotent matrix.

1. Introduction. Thisiis a continuation of three papers [8], [9] and [10]. A class of semigroups $M_n\{0, 1\}$ considered in [5] and [11] is considered as a part of a class of fuzzy matrix semigroups $M_n(F)$ (see [2], [3], [4]), where F is a finite set. A class of fuzzy matrix semigroups ([2], [3], [4]) is considered as a part of a class of Boolean matrix semigroups $M_n(2^S)$ (see [8], [9], [10]) where S is an arbitrary set. Fuzzy matrix semigroups $M_n(F)$ have their applications (see [6], [7]) in Mathematical Economics. In this paper we study nearly zero idempotent Boolean matrices (see Definition) in the semigroup $M_n(2^S = K)$ of all Boolean matrices over K, where S is a set (see [8], [9], [10]).

2. Definition and theorem

We begin with a definition.

Definition Let S be a set and $K = 2^S$. We denote by $M_n(K)$ the semigroup (see [10]) of all $n \times n$ Boolean matrices over K.

$A = (a_{ij})$ in $M_n(K)$ is said to be a nearly zero Boolean idempotent matrix if $AA = A$, $a_{11} \neq \emptyset$ and $a_{ii} = \emptyset$ for all $i \geqq 2$, where $\emptyset$ denotes the empty set.

We prove the following theorem which characterizes nearly zero Boolean idempotent matrices in $M_n(K)$.

Theorem 1 $A = (a_{ij})$ is a nearly zero Boolean idempotent matrix iff $a_{11} \neq \emptyset$, $a_{ij} = a_{i1}a_{1j}$ for all i and j, and $a_{it_1}a_{t_1t_2}\cdots a_{t_ki} = \emptyset$ for $i \neq t_1$.

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Proof Suppose that the conditions hold for A. We show that A is a nearly zeor idempotent matrix. The last condition implies that $a_{ij}a_{ji}=\varnothing$ for $i\neq j$. Letting $B=(b_{ij})=AA$ we show that $b_{ij}=a_{ij}$ for all i and j. We first see that

$b_{ij}=a_{ij}$ for all i and j. We first see that $b_{11}=\sum_{t=1}^{n}a_{1t}a_{t1}=a_{11}a_{11}+a_{12}a_{21}+\cdots+a_{1t}a_{t1}$

$+\cdots+a_{1n}a_{n1}=a_{11}a_{11}=a_{11}$ because $a_{1k}a_{k1}=\varnothing$ for $k\neq 1$. The condition $a_{ij}=a_{i1}a_{1j}$ implies that $a_{1j}=a_{11}a_{1j}$ ($j\geqq 2$) and hence a_{1j} is a subset of a_{11}. Similarly, w we have that a_{j1} is a subset of a_{11} ($j\geqq 2$). Thus we can see that $b_{1j}=\sum_{t=1}^{n}a_{1t}a_{tj}$

$=a_{11}a_{1j}+a_{12}a_{2j}+\cdots+a_{1n}a_{nj}=a_{1j}+a_{12}a_{21}a_{1j}+\cdots+a_{1n}a_{n1}a_{1j}=a_{1j}$ because $a_{1u}a_{u1}=\varnothing$ ($u\geqq 2$). Similarly, we can prove that $b_{j1}=a_{j1}$ for all $j\geqq 2$. We now show that $b_{ij}=a_{ij}$ for all i and j greater than 1.

We see that $b_{ij}=\sum_{t=1}^{n}a_{it}a_{tj}=a_{ij}+\sum_{t=2}^{n}a_{it}a_{tj}=a_{ij}+\sum_{t=2}^{n}a_{i1}a_{1t}a_{t1}a_{1j}=a_{ij}$ because $a_{tj}=a_{t1}a_{1j}$ and $a_{1t}a_{t1}=a_{t1}a_{1t}=\varnothing$ for $t>1$. Thus we have proved that $b_{ij}=a_{ij}$ for all i and j. (We note that $a_{ii}=a_{i1}a_{1i}=\varnothing$ for $i>1$.) We have that A is a nearly zero Boolean idempotent matrix. Conversely we assume that A is a nearly zero idempotent matrix. For $k>1$, we have that $\varnothing=a_{kk}=\sum_{t=1}^{n}a_{kt}a_{tk}$ and hence $a_{kt}a_{tk}=\varnothing$ for all t and $k>1$. From $A^{k-1}=A$, it is not difficult to show that, for $i\neq t_1$, $a_{it_1}a_{t_1t_2}\cdots a_{t_{k-1}t_k}a_{t_ki}=\varnothing$ which is a term of $a_{ii}=\sum_{t_k}\cdots\sum_{t_2}\sum_{t_1}a_{it_1}a_{t_1t_2}\cdots a_{t_{k-1}t_k}a_{t_ki}$. (1) It is clear that $a_{11}=a_{11}a_{11}$ and $a_{ii}=a_{i1}a_{1i}$ for $i>1$. We show that $a_{ij}=a_{i1}a_{1j}$ for $i\geqq 2$, $j\geqq 2$, in several steps. We define $a_{ij}(2)=\sum_{t=1}^{n}a_{it}a_{tj}$ and $a_{ij}(2:t)=a_{it}a_{tj}$. Then we see that $a_{ij}(2:s)=\varnothing$ for $s\in\{i,j\}$. We know that $a_{ij}(2:1)=a_{i1}a_{1j}$ is to be proven. Thus we may say that $a_{ij}(2)$ has $(n-3)$ terms to be considered. Letting $k\neq 1$, we assume that $a_{ij}(2:k)$ is one of $(n-3)$ terms of $a_{ij}(2)$. We shall show that $a_{ij}(2:k)=\varnothing$, which showing that $a_{ij}=a_{i1}a_{1j}$. (2) We define $a_{ij}(3)=$

$\sum_{t=1}^{t}a_{ik}a_{kt}a_{tj}$ and $a_{ij}(m)=\sum_{t_{m-2}}\cdots\sum_{t_2}\sum_{t_1}a_{ik}a_{t_1t_2}\cdots a_{t_{m-3}t_{m-2}}a_{t_{m-2}j}$ for $m>3$. We can see that $a_{ij}(3)$ has $(n-3)$ terms because $a_{ij}(3:s)=a_{ik}a_{ks}a_{sj}=\varnothing$ for all s in $\{i,k,j\}$. We denote $\{i,j,k\}$ by $T(3)$. Inductively we assume that $(m-3)a_{ij}(m-1)=\sum_{t_{m-3}}\cdots\sum_{t_2}\sum_{t_1}$. $a_{ik}a_{kt_1}a_{t_1t_2}\cdots a_{t_{m-3}j}$ has $(n-3)(n-4)\cdots(n-m+1)$ terms to be considered. We show that $a_{ij}(m)$ has $(n-3)(n-4)\cdots(n-m)$ terms (to be considered). (3) To prove this we define $a_{ij}(m:k_1,k_2,\cdots,k_{m-3},t)=a_{ik}a_{kk_1}a_{k_1k_2}\cdots a_{k_{m-3}t}a_{tj}$ and a set $T(m)=\{i,k,k_1,\cdots,k_{m-3},j\}$. We can prove that the cardinality $|T(m)|$ of the $T(m)$ is equal to

m, that is, $|T(m)| = m$. It is trivial to show that $a_{ij}(m: k_1, k_2, \cdots, k_{m-3}, s)$ for $s \in T(m)$. Thus we have proved that $a_{ij}(m)$ has $(n-3)(n-4)\cdots(n-m)$ terms. (4) We can have $m = n$, $a_{ij}(m) = \emptyset$ and consequently we have $a_{ij}(2: k) = \emptyset$. Thus we have $a_{ij} = a_{i1}a_{1j}$ for $i \neq 1 \neq j$. (If $n = 2$ or $n = 3$ then we can prove that $a_{ij} = a_{i1}a_{1j}$.)

(5) For $k \neq 1$ we can see that $a_{1k} = \sum_{t=1}^{n} a_{1t}a_{tk} = a_{11}a_{1k}$ because $a_{1t}a_{tk} = a_{1t}a_{t1}a_{1k} = \emptyset$ $(t \neq 1)$. similarly, we have $a_{kl} = a_{k1}a_{1l}$. This proves the theorem.

3. An Additional Theorem.

We shall prove the following theorem.

Theorem 2 Let $A = (a_{ij}) \in M_n(K)$ and assume that $a_{ii} \neq \emptyset$ for $i \leqq k_0$ and $a_{ii} = \emptyset$ for $i > k_0$, where k_0 is a positive integer such that $2 \leqq k_0 < n$. Then A is an idempotent matrix iff $a_{ij} = \sum_{t=1}^{k_0} a_{it}a_{tj}$ and $a_{it_1}a_{t_1t_2}\cdots a_{t_mj} \subseteq a_{ij}$ for all i and j.

Proof Suppose that the condition holds for A. Letting $AA = B(b_{ij})$ we show that $b_{ij} = a_{ij}$. We can see that $b_{ij} = \sum_{t=1}^{n} a_{it}a_{tj} = \sum_{t=1}^{k_0} a_{it}a_{tj} + \sum_{t=k_0+1}^{n} a_{it}a_{tj} = a_{ij} + \sum_{t=k_0+1}^{n} a_{it}a_{tj} = a_{ij}$ since $\sum_{t=k_0+1}^{n} a_{it}a_{tj} \subseteq a_{ij}$. Thus we have that $a_{ij} = b_{ij}$. Conversely we show that if A is an idempotent then $a_{ij} = \sum_{t=1}^{k_0} a_{it}a_{tj}$ and $a_{it_1}a_{t_1t_2}\cdots a_{t_mj} \subseteq a_{ij}$ for all i and j. We assume that A is an idempotent matrix. Then $AA\cdots A = A^{m+1} = A$, from which we obtain that $a_{it_1}a_{t_1t_2}\cdots\cdots a_{t_mj} \subseteq a_{ij}$. We show that $a_{ij} = \sum_{t=1}^{k_0} a_{it}a_{tj}$. We can see that $a_{kk} = \sum_{t=1}^{n} a_{kt}a_{tk} = \sum_{t=1}^{k_0} + \sum_{t=k_0+1}^{n} a_{kt}a_{tk} = \sum_{t=1}^{k_0} a_{kt}a_{tk}$ since $a_{kt}a_{tk} = a_{tk}a_{kt} = \emptyset$ for $t > k_0$. We define $a_{ij}(2) = \sum_{t=1}^{n} a_{it}a_{tj}$ and $a_{ij}(2: t) = a_{it}a_{tj}$ as a function of t as well as a term of $a_{ij}(2)$.

(1) Let $a_{ij}(2: k) = a_{ik}a_{kj}$ for $k > k_0$ and $i \neq j$. We shall show that $a_{ij}(2: k)$ is a subset of $\sum_{t=1}^{k_0} a_{it}a_{tj}$ in several steps. We note that $a_{ij}(3: k) = a_{ik}\left(\sum_{t=1}^{n} a_{kt}a_{tj}\right)$ has $n-1$ terms to be considered becauce $a_{ik}a_{kk}a_{kj} = a_{ij}(3: k, k) = \emptyset$. We define $a_{ij}(m: k_1, k_2, \cdots, k_{m-3}) = a_{ik}a_{kk_1}a_{k_1k_2}\cdots a_{k_{m-4}k_{m-3}}\left(\sum_{t=1}^{n} a_{k_{m-3}t}a_{tj}\right)$ assuming that $a_{ij}(m-1: k_1, k_2, \cdots, k_{m-4}) = a_{ik}a_{kk_1}a_{k_1k_2}\cdots a_{k_{m-5}k_{m-4}}\left(\sum_{t=1}^{n} a_{k_{m-4}t}a_{tj}\right)$ has $(n-(m-3))$ terms to be considered. (This means that the set $\{k_1, k_2, \cdots, k_{m-4}\}$ has the cardinality $m-4$ and each s in $\{k_1, k_2, \cdots, k_{m-4}\}$ is greater than k_0.)

(2) We prove that $a_{ij}(m: k_1, k_2, \cdots, k_{m-3})$ has $(n-(m-2))$ terms to be

considered. (We note that for a case $m=4$ a proof is simple and hence we assume that $m>4$.) To prove it we define $a_{ij}(m: k_1, k_2, \cdots, k_{m-3}, t)=a_{ik}a_{kk_1}a_{k_1k_2}\cdots a_{k_{m-3}t}a_{ti}$ as a function of t as well as a term of $a_{ij}(m: k_1, k_2, \cdots, k_{m-3})$. It is clear that $a_{ij}(m: k_1, k_2, \cdots, k_{m-3}, s)=\varnothing$ for s in $\{k, k_1, k_2, \cdots, k_{m-3}\}$ which has the cardinality $m-2$ because of the assumption on the set in (1). Thus we have shown that $a_{ij}(m: k_1, k_2, \cdots, k_{m-3})$ has $(n-(m-2))$ terms to be considered.

(3) We know that $a_{it_1}a_{t_1t_2}\cdots a_{t_mj}\subseteq a_{ij}$ and hence $a_{ik}(\sum_{t=1}^{k_0} a_{kt}a_{tj})\subseteq \sum_{t=1}^{k_0} a_{it}a_{tj}$. Therefore when we take out terms $a_{ik}(\sum_{t=1}^{k_0} a_{kt}a_{tj})$ from our counting we may say that $a_{ij}(3:k)$ has $n-(k_0+1)$ terms to be considered. Similarly, we can say that $a_{ij}(m: k_1, k_2, \cdots, k_{m-3})$ has $(n-(k_0+m-2))$ terms to be considered. If $n-(k_0+m-2)=0$ then there are no terms of $a_{ij}(m: k_1, k_2, \cdots, k_{m-3})$ to be considered.

(4) We conclude that $a_{ij}(2:k)$ is a subset of $\sum_{t=1}^{k_0} a_{it}a_{tj_{k_0}}$ from (1), (2) and (3). Thus we have proved that $a_{ij}=\sum_{t=1}^{k_0} a_{it}a_{tj}$. This proves the theorem.

We state the following

Proposition Let $A=(a_{ij})\in M_n(K)$. If A is an idempotent matrix and $a_{ii}=\varnothing$ for all i, then $a_{ij}=\varnothing$ for all i and j.

A technique of ihe proof or the proposition is similar to that of the proof of Theorem 1 and we omit the proof.

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