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Score Vectors of t-reducible Tournaments*

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Abstract The definitions of t reducible and exactly t reducible n tournaments are introduced. Criteria are found for determining (i) whether a tournament with a given score vector R is t-reducible and (ii) whether it is exactly t reducible.

I. Iutroduction

Let T_n be an n tournament with vertex set $V(T_n)=\{v_1, v_2, \cdots, v_n\}$. The score of the vertex v_i in T_n is denoted by r_i, $i=1,2,\cdots,n$ and $r_1 \leq r_2 \leq \cdots \leq r_n$. Then the score vector of T_n is $R=(r_1, r_2, \cdots, r_n)$. Conversely, let $R=(r_1, r_2, \cdots, r_n)$ be an non negative integral vector in which $0 \leq r_1 \leq r_2 \leq \cdots \leq r_n$, then R is said to be a score vector if there is some T_n such that its score vector is just R. The set of all n tournaments with score vector R is denoted by $\mathcal{T}(R)$. H. G. Landau [1] has shown the following.

Theorem 1.1 Let $R=(r_1, r_2, \cdots, r_n)$ be an non negative integral vector in which $0 \quad r_1 \quad r_2 \quad \cdots \quad r_n$. Then $\mathcal{T}(R) \neq \phi$ if and only if for $j=1,2,\cdots,n$,

$$r_1+r_2+\cdots+r_j \geq \binom{j}{z},$$

with equality for $j=n$.

For strong tournaments, F. Harary and L. Moser [2] have proved the follwing.

Theorem 1.2 Let $R=(r_1, r_2, \cdots, r_n)$ be the score vector of T_n. Then T_n is strong if and only if for $j=1,2,\cdots,n-1$,

$$r_1+r_2+\cdots+r_j \geq \binom{j}{2}+1.$$

In this paper, we introduce the new definition as follows.

Definition 1.3 An n tournament T_n is called t reducible if its every sub tournament induced by $n-t+1$ vertices of T_n is reducible. A t reducible n tournament T_n is said to be exactly t reducible if it is not $(t+1)$ reducible.

It is easy to see by Definition 1.3 that a 1 reducible n tournament is reducible. Considering (degenerate) tournaments with one vertex to be strong and

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observing that no tournament with two vertices is strong, it follows that (1) no n tournament is n reducible, (2) every n tournament is exactly $(n-1)$ reducible and (3) an n tournament can be $(n-2)$ reducible, but it cannot be exacty $(n-2)$ reducible.

The purpose of this paper is give criteria for determining whether a given n tournament wilh score vector R is (i) t reducible and (ii) exactly t reducible.

In what follows, we always assume that T_n U denote the subtournament which is obtained by deleting the set $U \subseteq V(T_n)$ from T_n. Particularly, we write T_n u for T_n U if $U = \{u\}$.

2. Mian Results

Lemma 2.1 Let $2 \leqslant t \leqslant n-2$. Then the t reducible n tournament T_n is $(t-1)$ reducible.

Proof Assume T_n is not $(t-1)$ reducible. Then T_n has a strong subtournament T_{n-t+2} induced by $n-t+2$ vertices of T_n. Observing $n-t+2 \geqslant 4$, T_{n-t+2} has an $(n-t+1)$ cycle. Let the vertex u of T_{n-t+2} be not contained in the cycle. Then the subtournament $T_{n-t+2}-u$ is strong. Hence T_n will be not t reducible, a contradiction. Thus, T_n must be $(t-1)$ reducible. The proof is completed.

Corollary 2.2 Let $1 \leqslant t \leqslant n-2$. Then the t reducible n tournament T_n is reducible.

Proof It is a simple consequence of Lemma 2.1.

Suppose the n tournament T_n is reducible. Then T_n has a decomposition into the strong components $S_1, S_2, \cdots, S_p$, $p \geqslant 2$. such that $V(S_1)$, $V(S_2)$, $\cdots$, $V(S_p)$ is a non trivial partition of $V(T_n)$ and $v \to u$ for any $u \in V(S_i)$ and $v \in V(S_j)$, where $1 \leqslant i < j \leqslant p$. On the decomposition $S_1, S_2, \cdots, S_p$ of T_n, we have.

Lemma 2.3 Let T_n has a decomposition into strong components $S_1, S_2, \cdots$, S_p, $p \geqslant 2$ and let $n_i = |V(S_i)|$, $i = 1, 2, \cdots, p$. Then, for $i = 1, 2, \cdots, p$,

$$V(S_i) = \{v_{k_{i-1}+1}, v_{k_{i-1}+2}, \cdots, v_{k_i}\},$$

where $k_0 = 0$ and $k_i = n_1 + n_2 + \cdots + n_i$, $i = 1, 2, \cdots, p$.

Proof Let $r(u)$ be the score of $u \in V(T_n)$. Since S_1, S_2, $\cdots$, S_p is a decomposition of T_n, so that $v \to w$ for any $w \in V(S_i)$ and $v \in V(S_j)$, $1 \leqslant i < j \leqslant p$. If S_i is trivial, i.e., $n_i = 1$, then

$$r(w) = n_1 + n_2 + \cdots + n_{i-1} < n_1 + n_2 + \cdots + n_{i-1} + n_i + \cdots + n_{j-1} \leqslant r(v);$$

If S_i is non trivial, then $n_i \geqslant 3$ and the score $\tilde{r}(w)$ of w in S_i satisfies $\tilde{r}(w) \leqslant n_i - 2$. Therefore,

$$r(w) \leqslant n_1 + n_2 + \cdots + n_{i-1} + (n_i - 2) < n_1 + n_2 + \cdots + n_i \leqslant r(v).$$

Thus, the vertices with smaller index in $V(T_n)$ are contained in the component with smaller index. The lemma is proved.

Let $R = (r_1, r_2, \cdots, r_n)$ be the score vector of T_n. Denote

$$h_j = r_1 + r_2 + \cdots + r_j - \binom{j}{2}, \quad j = 1, 2, \cdots, n-1.$$

Let

$$h(\mathbf{R}) = \min\{h_j \mid j = 1, 2, \cdots, n-1\}.$$

By Theorem 1.2, T_n is reducible if and only if $h(\mathbf{R}) = 0$. Write

$$J = \{j \mid 1 \leq j \leq n \text{ and } h_j = 0\}.$$

From Theorem 1.1 and 1.2, T_n is reducible if and only if $|J| \geq 2$, where $|J|$ is the cardinal of J. Let T_n be reducible and denote $J = \{j_1, j_2, \cdots, j_m\}$ in which $0 = j_0 < j_1 < j_2 < \cdots < j_{m-1} < j_m = n$. The vector $(j_0, j_1, j_2, \cdots, j_{m-1}, j_m)$ is called the reducible type of T_n. Clearly, the reducible type of n tournaments depends on only its score vector **R**.

Lemma 2.4 The reducible type of T_n is $(j_0, j_1, j_2, \cdots, j_m)$ if and only if T_n has a decomposition $S_1, S_2, \cdots, S_m$ such that

$$V(S_i) = \{v_{j_{i-1}+1}, v_{j_{i-1}+2}, \cdots, v_{j_i}\},$$

where $i = 1, 2, \cdots, m$.

Proof Suppose that the reducible type of T_n is $(j_0, j_1, j_2, \cdots, j_m)$, then

$$r_1 + r_2 + \cdots + r_{j_i} = \binom{j_i}{2}, \quad i = 1, 2, \cdots, m. \qquad (2.1)$$

and for $j \neq j_i$, $i = 1, 2, \cdots, m$,

$$r_1 + r_2 + \cdots + r_j \geq \binom{j}{2} + 1. \qquad (2.2)$$

Let S_i denote the subtournament induced by the vertices $v_{j_{i-1}+1}, v_{j_{i-1}+2}, \cdots, v_{j_i}$ in T_n. By (2.1) and (2.2), the score $\tilde{r}_j$ of $v_{j_{i-1}+j}$ is equate to $r_{j_{i-1}+j} - j_{i-1}$. Therefore, for $j = 1, 2, \cdots, j_i - j_{i-1} - 1$,

$$\tilde{r}_1 + \tilde{r}_2 + \cdots + \tilde{r}_j = \sum_{k=1}^{j} r_{j_{i-1}+k} - j \cdot j_{i-1} = \sum_{k=1}^{j_{i-1}+j} r_k - \sum_{k=1}^{j_{i-1}} r_k - j \cdot j_{i-1}$$

$$\geq \binom{j_{i-1}+j}{2} + 1 - \binom{j_{i-1}}{2} - j \cdot j_{i-1} = \binom{j}{2} + 1.$$

By Theorem 1.2, S_i is strong, $i = 1, 2, \cdots, m$. Moreover, let T_{j_i} be the subtournament induced by the vertices $v_1, v_2, \cdots, v_{j_i}$. If $u \in V(S_i)$ and $v \in V(S_j)$, $1 \leq i < j \leq m$, then $u \in V(T_{j_i})$ and $v \in V(T_n - T_{j_i})$. By (2.1) and (2.2), $v \rightarrow u$. Thus, $S_1, S_2, \cdots, S_m$ is a decomposition of T_n.

Conversely, suppose that T_n has a decomposition $S_1, S_2, \cdots, S_m$ such that $V(S_i) = \{v_{j_{i-1}+1}, v_{j_{i-1}+2}, \cdots, v_{j_i}\}$, $i = 1, 2, \cdots, m$. Then the score $\tilde{r}_j$ of $v_{j_{i-1}+j}$ in S_i is equale to $r_{j_{i-1}+j} - j_{i-1}$, $j = 1, 2, \cdots, j_i - j_{i-1}$. Observing S_i is strong, it follows from Theorem 1.2 that for $j = 1, 2, \cdots, j_i - j_{i-1} - 1$,

$$\tilde{r}_1 + \tilde{r}_2 + \cdots + \tilde{r}_j = \sum_{k=1}^{j} r_{j_{i-1}+k} - j j_{i-1} \geq \binom{j}{2} + 1,$$

and

$$\tilde{r}_1+\tilde{r}_2+\cdots+\tilde{r}_{j_i-j_{i-1}}=\sum_{k=1}^{j_i-j_{i-1}} r_{j_{i-1}+k}-(j_i-j_{i-1})j_{i-1}=\binom{j_i-j_{i-1}}{2}.$$

Therefore,

$$\sum_{k=1}^{j_i-j_{i-1}} r_{j_{i-1}+k}=\binom{j_i-j_{i-1}}{2}+(j_i-j_{i-1})j_{i-1}=\binom{j_i}{2}-\binom{j_{i-1}}{2}$$

Thus,

$$\sum_{k=1}^{j_i} r_k=\sum_{k=1}^{j_1} r_k+\sum_{k=1}^{j_2-j_1} r_{j_1+k}+\cdots+\sum_{k=1}^{j_i-j_{i-1}} r_{j_{i-1}+k}$$
$$=\binom{j_1}{2}+\left[\binom{j_2}{2}-\binom{j_1}{2}\right]+\cdots+\left[\binom{j_i}{2}-\binom{j_{i-1}}{2}\right]=\binom{j_i}{2}.$$

Moreover, if $j\neq j_i, i=1,2,\cdots,m$, then there is some integer j_{i_0} such that $j_{i_0-1}<j<j_{i_0}$. Thus,

$$\sum_{k=1}^{j} r_k=\sum_{k=1}^{j_{i_0-1}} r_k+\sum_{k=1}^{j-j_{i_0-1}} r_{j_{i_0-1}+k}$$
$$\geqslant\binom{j_{i_0-1}}{2}+(j-j_{i_0-1})j_{i_0-1}+\binom{j-j_{i_0-1}}{2}+1\geqslant\binom{j}{2}+1.$$

This shows that the reducible type of T_n is $(j_0, j_1, j_2, \cdots, j_{m-1}, j_m)$

The proof of the lemma is completed.

Denote

$$a(\mathbf{R})=\max\{a_i=j_i-j_{i-1} \mid i=1,2,\cdots,m\},$$

where $(j_0, j_1, j_2, \cdots, j_m)$ is the reducible type of T_n with score vector R. By Lemma 2.4, $a(\mathbf{R})$ is the size of the largest (i.e. greatest number of vertices) strong component of $T_n\in\mathscr{T}(\mathbf{R})$.

The mian result of this paper is the following .

Theorem 2.5 Let $\mathbf{R}=(r_1,r_2,\cdots,r_n)$ be a score vector. Then $T_n\in\mathscr{T}(\mathbf{R})$ is exactly t reducible if and only if $a(\mathbf{R})=n-t$, where $1\leqslant t\leqslant n-3$.

Proof To prove the theorem, it suffices to show that T_n is exactly t reducible if and only if the size of the laregest component of T_n is of size $n-t$.

Suppose T_n is exactly t reducible. Then, (1) T_n is t-reducible and (2) T_n is not $(t+1)$ reducible. From (1), every subtournament with at least $n-t+1$ vertices is reducible Thus, there is no component of size greater than $n-t$. From (2), T_n has a component of size at least $n-t$. Thus the largest component is of size $n-t$.

Couversely, suppose the largest component of T_n is of size $n-t$. Then T_n has a strong subtournament with $n-t$ vertices. Thus T_n is not $(t+1)$-reducible. Moreover, T_n has a decomposition $S_1, S_2, \cdots, S_p$, $p\geqslant 2$ such that for some i, $|V(S_i)|=n-t$. Let T_{n-t+1} be any subtournament induced by $n-t+1$ vertices of T_n. Clearly, there are Some j such that $V(T_{n-t+1})\cap V(S_j)\neq\phi$. Denote

$$j_0 = \max\{ j \mid 1 \leqslant j \leqslant p \text{ and } V(T_{n-t+1}) \cap V(S_j) \neq \phi \}$$

Write $U = V(T_{n-t+1}) \cap V(S_{j_0})$. Obviously, $1 \leqslant |U| \leqslant |V(S_{j_0})| \leqslant |V(S_i)| = n-t$. By $|V(T_{n-t+1})| = n-t+1$. so that $W = V(T_{n-t+1} - U) \neq \phi$. Thus U and W is an non trivial partition of $V(T_{n-t+1})$. Let $w \in W$ and $u \in U$, then for some j, $w \in V(S_j)$ and $u \in V(S_{j_0})$. By the maximality of j_0, it follows that $j \leqslant j_0$. Considering $S_1, S_2, \cdots, S_p$ is a decomposition of T_n, so that $u \to w$. This implies that T_{n-t+1} is reducible. Thus T_n is t-reducible. This proves that T_n is exactly t-reducible.

This is complete the proof of the theorem.

Theorem 2.6 The n-tournament T_n is $(n-2)$-reducible if and only if it is transitive, where $n \geqslant 3$.

Proof Obvious.

By using Theorem 2.5 and 2.6, it follows that every n-tournament in $\mathscr{T}(R)$ is exactly t-reducible if $\mathscr{T}(R)$ has an exactly t-reducible n tournament.

Corollary 2.7 Let $n \geqslant 3$. Then $T_n \in \mathscr{T}(R)$ is t reducible if and only if $a(R) \leqslant n-t$, where $1 \leqslant t \leqslant n-2$.

Proof This a simple consequence of Theorem 2.5 and 2.6.

Corollary 2.8 Let $n \geqslant 4$. Then $T_n \in \mathscr{T}(R)$ is exactly 1 reducible if and only if one of the following conditions holds:

(1) $r_1 = 0$ and for $j = 2, 3, \cdots, n-1$, $r_1 + r_2 + \cdots + r_j \geqslant \binom{j}{2} + 1$,

(2) $r_n = n-1$ and for $j = 1, 2, \cdots, n-2$, $r_1 + r_2 + \cdots + r_j \geqslant \binom{j}{2} + 1$.

Proof This is the case $a(R) = n-1$ in Theorem 2.5.

Remark Corollary 2.8 gives a criteia for determining whether a n tournament has no Hamiltonian cycle, but it has an $(n-1)$-cyde.

Corollary 2.9 Let $n \geqslant 4$. Then $T_n \in \mathscr{T}(R)$ is reducible if and only if one of the following conditions holds:

(1) $r_1 = 0$ and $r_n = n-1$;

(2) there is some integer j, $2 \leqslant j \leqslant n-2$ such that

$$r_1 + r_2 + \cdots + r_j = \binom{j}{2}.$$

Proof This is the case $t = 2$ in Corollary 2.7.

References

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