J. Math. Res. & Exposition, 1988, No.4

Second-Order Expansion for a Class of Quasidifferentiable Functions*

Z. Q. Xia (夏尊铨)
(Dalian University of Technology, China)

Suppose f, $g: \mathbf{R}^n \to \mathbf{R}$ are quasidifferentiable, and for an interval $[x, y] \subset \mathbf{R}^n$ one has $0 \bar{\in} \langle \underline{\partial} g(\xi) + \bar{\partial} g(\xi), y-x \rangle$, $\xi \in (x, y)$. Then there exists an $\eta \in (x, y)$ such that

$$\frac{f(y)-f(x)}{g(y)-g(x)} \in \frac{\langle \underline{\partial} f(\eta) + \bar{\partial} f(\eta), y-x \rangle}{\langle \underline{\partial} g(\eta) + \bar{\partial} g(\eta), y-x \rangle}$$

Suppose a function f defined in $\mathbf{R}^n$ is differentiable and its gradient is quasidifferentiabe. Then for any $x \in \mathbf{R}^n$, $d \neq 0$, and $\lambda > 0$, one has $f(x+\lambda d) \in f(x) + \lambda f'(x; d) + (\lambda^2/2) d^T (\underline{\partial}^2 f(x+\xi d) + \bar{\partial}^2 f(x+\xi d)) d$, for some $\xi \in (0, 1)$.

Let $f: \mathbf{R}^n \to \mathbf{R}^m$ where $f = (f_1, \cdots, f_m)^T$ and $\varphi: \mathbf{R}^m \to \mathbf{R}^n$. Suppose f and φ are continuously differentiable, and assume that $\nabla f(x)$ is quasidifferentiable and $\nabla \varphi$ is uniformly quasidifferentiable. Then, (φ, f) is twice quasidifferentiable and one has that

$$(\varphi \cdot f)''(x; d_1, d_2) = (\varphi'_{f(x)} \cdot f''_x + \varphi''_{f(x)} \cdot [f'_x])(d_1, d_2),$$

where $[f'_x](d_1, d_2) := (f'_x(d_1), f'_x(d_2))^T = (f'(x; d_1), f'(x; d_2))^T$,

and

$$D^2(\varphi \cdot f)(x) = \nabla_f^T \varphi(f(x)) \cdot D^2 f(x) + J^T f(x) \cdot D_f^2 \varphi(f(x)) J f(x),$$

where
$$D^2 f := \begin{vmatrix} D^2 f_1 \\ \vdots \\ D^2 f_m \end{vmatrix},$$

$$J^T f(x) D_f^2 \varphi(f(x)) J f(x) = [J^T f(x) \cdot \underline{\partial}_f^2 \varphi(f(x)) J f(x), J^T f(x)(\bar{\partial}_f^2 \varphi(f(x))) J f(x)].$$

Under some assumptions suppose x is a local minimum point to minimize $f(x)$, $x \in \mathbf{R}^n$. Then for each pair of $V \in \underline{\partial}^2 f(x)$ and $W \in \bar{\partial}^2 f(x)$ there exist vectors $g_1, g_2 \in B_1(0)$ such that

$$g_2^T V - g_1^T W \in g_1^T \underline{\partial}^2 f(x) - g_2^T \bar{\partial}^2 f(x),$$

where $g_1 = \Sigma \lambda_i d_i^{+T}$, $g_2 = \Sigma \lambda_i d_i^{-T}$, $d_i \in \mathrm{bd} B_1(0)$, $\lambda_i \geq 0$, $i = 1, \cdots, m$, $\Sigma \lambda_i = 1$.

The condition given above can be rewritten as

$$(\Sigma \lambda_i d_i^{-T} V - \Sigma \lambda_i d_i^{+T} W) \in (\Sigma \lambda_i d_i^{+T}) \underline{\partial}^2 f(x) - (\Sigma \lambda_i d_i^{-T}) \bar{\partial}^2 f(x),$$

or

$$-\Sigma \lambda_i d_i^T \diamond_2 (V, W) \in \Sigma \lambda_i d^T \diamond_1 D^2 f(x).$$

* WP 87 068, July, IIASA, Laxenburg Austria.