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The Oscillation and Nonoscillation Theorems for A Class of General Second Order Functional Differential Equations *

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Abstract

The purpose of this paper is to give some sufficient conditions on f and g for ensuring that all solutions or all bounded solutions of general second order functional differential equation

$$[r(t)g(y'(t))]' + f(t, y(t), y(p(t)), y'(t), y'(q(t))) = 0$$

are oscillatory, or it has at least one bounded nonoscillatory solution.

I. Introduction

The paper deals with the oscillation and nonoscillation of a class of general second order functional differential equations

$$[r(t)g(y'(t))]' + f(t, y(t), y(p(t)), y'(t), y'(q(t))) = 0 \qquad (1)$$

where $r(t)$, $p(t)$, $q(t)$ are continuous functions mapping $R_a = [a, \infty)$ into R, $r(t) > 0$ if $t \geq a$ and $p(t) \to \infty$, $q(t) \to \infty$ as $t \to \infty$, g: $R \to R$, is a continuous increasing function, $g(0) = 0$, $f(t, u, v, w, z)$ mapping $R_a \times R^4$ into R is continuous.

In this paper, we assume that every solution y of (1) exists in $[t_y, \infty)$, where $t_y \geq a$. A solution $y(t)$ is called oscillation if its zero points is unbounded in $[t_y, \infty)$.

Equation (1) is a class of general equation, it contains the equation

$$[r(t)y'(t)]' + f(t, y(t), y(p(t)), y'(t), y'(q(t))) = 0$$

which was dealed with by many papers.

The purpose of this paper are to give some sufficient conditions on f and g for ensuring that all solutions or all bounded solutions of (1) are oscillatory, or (1) has at least one bounded nonoscillatory solution.

II. Oscillation

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Theorem 1. Suppose that the following conditions hold for $t \geq a > 0$:

(i) $uf(t, u, v, w, z) > 0$ as $uv > 0$,

(ii) if $y(t)$ is a positive increasing (or negative decreasing) function, then

$$f(t, y(t), y(p(t)), y'(t), y'(q(t)))dt = \infty \text{ (or } -\infty),$$

(iii) for any constant $c > 0$, we have

$$\int^{\infty} g^{-1}\left(\frac{\pm c}{r(t)}\right) dt = \pm\infty,$$

then every solution of (1) is oscillatory.

Proof. If the conclusion is not true, there exits a nonoscillatory solution $y(t)$ of (1). We may assume that $y(t) > 0$ and $y(p(t)) > 0$ for $t \geq T \geq a$ without loss of generality. From (i), we have $f(t, y(t), y(p(t)), y'(t), y'(q(t))) > 0$ for $t \geq T$ and then $[r(t)g(y'(t))]' < 0$ and $r(t)g(y'(t))$ is decreasing for $t \geq T$, therefore, we have tow cases to discuss:

Case 1. $r(t)g(y'(t)) > 0$ for $t \geq T$.

According to $r(t) > 0$ and $g(y'(t)) > 0$ for $t \geq T$, we know that $y'(t) > 0$ for $t \geq T$ and $y(t)$ is positive increasing in $[T, \infty)$, Integrating (1) from T to $t > T$ we have

$$r(t)g(y'(t)) - r(T)g(y'(t)) = -\int_T^t f(s, y(s), y(p(s)), y'(s), y'(q(s)))ds.$$

Notice condition (ii), we get $r(t)g(y'(t)) < 0$ for t sufficiently large from above formula. This is a contradiction.

Case 2. There exists $T_1 > T$ such that $r(t)g(y'(t)) < 0$ for $t \geq T_1$.

Since $r(t)g(y'(t))$ decrease, we get

$$r(t)g(y'(t)) < r(T_1)g(y'(T_1)) \text{ for } t > T_1.$$

Then,

$$y'(t) < g^{-1}\left(\frac{r(T_1)g(y'(T_1))}{r(t)}\right),$$

$$y(t) < y(T_1) + \int_{T_1}^t g^{-1}\left(\frac{r(T_1)g(y'(T_1))}{r(s)}\right) ds.$$

From (iii), we get $y(t) < 0$ for t sufficiently large. This is a contradiction to $y(t) > 0$.

If $y(t) < 0$ for $t \geq T$, we can get a similar argument. This completes the proof.

Theorem 1 generalize the theorem 1 of paper [1] beeause the condition III of Theorem 1 in [1] is equivalent to the condition (ii).

Theorem 2. Suppose that the condition (i) and (ii) of Theorem 1 hold, $r(t)$ and $g(x)$ are differentiable, $r'(t) < 0$ for $t \geq a$, $g'(x) > 0$, then every solution of (1) is osillatory.

Proof. If $y(t)$ is a nonoscillatory solution of (1), let $y(t)>0$, $y(p(t))>0$ for $t\geq T\geq a$. Then $[r(t)g(y'(t))]'<0$ and $r(t)g(y'(t))$ is decreasing for $t\geq T$. Now, there are two cases to discuss as Theorem 1.

Case 1. $r(t)g(y'(t))>0$ for $t\geq T$.

It follows that $y'(t)>0$ and $y(t)$ is positive increasing for $t\geq T$. As the argument in Theorem 1 we can get $y'(t)<0$ for $t\geq T$. This derives a contradiction to case 1.

Case 2. There exist $T_1\geq T$ such that $r(t)g(y'(t))<0$ for $t\geq T_1$.

It follows that $y'(t)<0$ for $t\geq T_1$. From $[r(t)g(y'(t))]'<0$ we have

$$r'(t)g(y'(t))+r(t)g'(y'(t))y''(t)<0 \quad (t\geq T_1).$$

Then,

$$y''(t)<-\frac{r'(t)g(y'(t))}{r(t)g'(y'(t))} \qquad (t\geq T_1).$$

Consider $r'(t)<0$ and $g'(x)>0$, we get $y''(t)<0$ for $t\geq T_1$. It follows that $y(t)$ is decreasing and its curve is convex. So $y(t)$ is decreasing and its curve is convex. So $y(t)$ must be negative for t is sufficiently large. This contradicts $y(t)>0$.

If $y(t)<0$ for $t\geq T\geq a$, we can get the similar contradiction.

Theorem 3. If the following conditions hold:

(i) $uf(t,u,v,w,z)$ 0 if $uv>0$, $t\geq a$;

(ii) $r(t)\to\infty$ as $t\to\infty$ and $\int^{\infty} g^{-1}\left(\frac{c}{r(s)}\right)ds=\infty\cdot\operatorname{sgn} c$, where c is a arbitrary constant;

(iii) for any positive increasing function $y(t)$ and any constant $c>0$, or for any negative decreasing function $y(t)$ and any constant $c<0$, we have

$$\int_T^{\infty}(t-T)f(t,y(t),y(p(t)),y'(t)(q(t)))dt>c\int_T r(t)dt.$$

Then every solution of (1) oscillates.

Proof. Suppose that $y(t)>0$, $y(p(t))>0$ for $t\geq T\geq a$, then $r(t)gy(t))$ is decreasing for $t\geq T$. There are two cases:

Case 1. $r(t)g(y'(t))>0$ for $t\geq T$.

In this case, we get that $y'(t)>0$ and $y(t)$ is increasing for $t\geq T$. Integrating (1) form $\lambda\geq T$ to $t>\lambda$ we get

$$r(t)g(y'(t))-r(\lambda)g(y'(\lambda))+\int_\lambda^t f(s,y(s),y(p(s)),y'(s),y'(q(s)))ds=0$$

Then $r(\lambda)g(y'(\lambda))>\int_\lambda^t f(s,y(s),y(p(s)),y'(s),y'(q(s)))ds,\qquad (2)$

Integrate the both sides of (2) form T to $t<T$ and by part, we have

$$\int_T^t r(\lambda)g(y'(\lambda))d\lambda > \int_T^t (\lambda-T)f(\lambda, y(\lambda), y(p(\lambda)), y'(\lambda), y'(q(\lambda)))d\lambda \tag{3}$$

On the other hand, because $r(t)g(y'(t))$ is positive and decreasing, the limit $\lim_{t\to\infty} r(t)g(y'(t))$ is existent and finite, Notice condition (ii) $\lim_{t\to\infty} r(t)=\infty$, we have $\lim_{t\to\infty} g(y'(t))=0$, then $\lim_{t\to\infty} y'(t)=0$. Therefore, for any constant $c>0$, there exists $T_0>T$ such that $y'(t)<a$ for $t\geq T_0$ and then $g(y'(t))<g(a)$ for $t\geq T_0$. From (3) we have

$$g(a)\int_{T_0}^t r(\lambda)d\lambda > \int_{T_0}^t (\lambda-T)f(\lambda, y(\lambda), y(p(\lambda)), y'(\lambda), y'(g(\lambda)))d\lambda.$$

this contradictec condition (iii).

Case 2. $r(t)g(y'(t))<0$ for $t\geq T_1\geq T$.

Using the argument in Theorem 1 we can get a contradiction.

If $y(t)$ is eventually negative we can use the similar argument to discuss.

The proof is completed.

Theorem 4 Suppose that the conditions (i) and (iii) of Theorem 1 hold. In addition, we assume that (ii)′ if $y(t)$ is a positive increasing function (or a negative decreasing function), then

$$\int^\infty g^{-1}\left[\frac{1}{r(t)}\int_\lambda^t f(s, y(s), y(p(s)), y'(s), y'(q(s)))ds\right]d\lambda=\infty \ (\text{or} -\infty).$$

Then every bounded solution of (1) is oscillatory.

Proof. Let $y(t)$ is a bounded nonoscillatory solution of (1) and $y(t)>0$, $y(p(t))>0$ for $t\geq T\geq a$.

Then $r(t)g(y'(t))$ is decreasing for $t\geq T$. There are two cases to discuss:

Case 1. $r(t)g(y'(t))>0$ for $t\geq T$.

In this case, $y'(t)>0$ and $y(t)$ is positive increasing for $t\geq T$. From (2) we have

$$y'(\lambda)>g^{-1}\left[\frac{1}{r(\lambda)}\int_\lambda^t f(s, y(s), y(p(s)), y'(s), y'(q(s)))ds\right], \lambda\geq T,\ t\geq\lambda.$$

Then

$$y(\bar{t})-y(T)>\int_T^{\bar{t}} g^{-1}\left[\frac{1}{r(\lambda)}\int_\lambda^t f(s, y(s), y(p(s)), y'(s), y'(q(s)))ds\right]d\lambda,\ \bar{t}\geq T.$$

By the formula as above and condition (ii) we get that $y(\bar{t})\to\infty$ as $\bar{t}\to\infty$. This is a contradiction to the boundeduess of $y(t)$.

Case 2. $r(t)g(y'(t))<0$ for $t\geq T_1\geq T$.

For this case, we can get a contradiction by the argument in theorem 1.

We can prove that there exists a contradiction if $y(t)$ eventually negative.

Ⅲ. Nonoscillation

Theorem 5. If the following conditions hold for $t \geq T \geq a$:

(i) $uf(t,u,v,w,z) > 0$ if $uv > 0$;

(ii) $f(t,u,v,w,z)$ is increasing with respect to u, v, w, z if u, v, w, z are nonnegative;

(iii) for any $a > 0$ and $b \geq 0$, we have

$$g^{-1}\left[\frac{1}{r(t)}\int_t^{\infty} f(s,a,a,b,b)\,ds\right] \to 0 \quad \text{as} \quad t \to \infty.$$

(iv) for any $a > 0$, we have

$$\int_T^{\infty} g^{-1}\left[\frac{1}{r(t)}\int_t^{\infty} f(s,a,a,o,o)\,ds\right]dt < \infty.$$

Then (1) has at least a bounded increasing solution.

Proof. Consider the integral equation

$$y(t) = 1 - \int_t^{\infty} g^{-1}\left[\frac{1}{r(\lambda)}\int_\lambda^{\infty} f(s, y(s), y(p(s)), y'(s), y'(q(s)))\,ds\right]d\lambda. \qquad (4)$$

Obviously, its solution satisfies (1).

Set $y_0(t) \equiv 1$

$$y_n(t) = 1 - \int_t^{\infty} g^{-1}\left[\frac{1}{r(\lambda)}\int_\lambda^{\infty} f(s, y_{n-1}(s), y_{n-1}(p(s)), y'_{n-1}(s), y'_{n-1}(q(s)))\,ds\right]d\lambda,$$
$$n = 1, 2, \cdots. \qquad (5)$$

Then

$$y_1(t) = 1 - \int_t^{\infty} g^{-1}\left[\frac{1}{r(\lambda)}\int_\lambda^{\infty} f(s,1,1,0,0)\,ds\right]d\lambda,$$

$$y'_1(t) = g^{-1}\left[\frac{1}{r(t)}\int_t^{\infty} f(s,1,1,0,0)\,ds\right].$$

From (iv), there exists $t_1 \geq a$ such that

$$\int_t^{\infty} g^{-1}\left[\frac{1}{r(\lambda)}\int_\lambda^{\infty} f(s,1,1,0,0)\,ds\right]d\lambda < \frac{1}{2} \quad \text{for} \quad t \geq t_1$$

It follows that

$$\frac{1}{2} \leq y_1(t) < 1 \quad \text{for} \quad t \geq t_1 \qquad (6)$$

From (iii), there exists $t_2 \geq a$ such that

$$0 < y'_1(t) < 1 \quad \text{for} \quad t \geq t_2 \qquad (7)$$

From (4), (5), (6), (7) and Condition (ii) we have

$$y_2(t) = 1 - \int_t^{\infty} g^{-1}\left[\int_\lambda^{\infty} f(s, y_1(s), y_1(p(s)), y'_1(s), y'_1(q(s)))\,ds\right]d\lambda$$

$$\leq 1-\int_t^{\infty} g^{-1}\left[\int_{\lambda}^{\infty} f\left(s, \frac{1}{2}, \frac{1}{2}, 0, 0\right) ds\right] d\lambda,$$

$$y_2'(t)=g^{-1}\left[\int_t^{\infty} f(s, y_1(s), y_1(p(s)), y_1'(s), y_1'(q(s))) ds\right]$$

$$\leq g^{-1}\left[\int_t^{\infty} f(s,1,1,1,1) ds\right].$$

From conditions (iii) and (iv), we know that there is $T>t_1$ such that

$$\frac{1}{2}\leq y_2(t)<1, \quad 0<y_2'(t)<1 \quad \text{for} \quad t>T.$$

By mathematical induction we have

$$\frac{1}{2}\leq y_n(t)<1, \quad 0<y_n'(t)<1, \quad n=1,2,\cdots.$$

Thus the sequence $\{y_n(t)\}$ is equicontinuoue and Uniformly bounded for $t\geq T$, and then there exists a convergent subsequence $\{y_{n_k}(t)\}$. Let its limit function is $y(t)$, then

$$\frac{1}{2}\leq y(t)\leq 1 \quad \text{and} \quad 0\leq y(t)\leq 1 \quad \text{for} \quad t\geq T.$$

Therefore, $y(t)$ is bounded increasing for $t\geq T$. Obviously $y(t)$ satisfies the equation (3) and the equation (1) for $t\geq T$. This completes the proof.

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