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Some Optimality Conditions for The C_M-embedded Problem with the Euclidean Norm*

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The main purpose of this paper is to present optimality conditions for the C_M embedded problem. Given a convex set K containing the origin as one of its interior points and a bounded feasible region S, this problem consists in finding the largest convx bodies contained in S, associated with the Minkowski function $m(x; K)$. This problem is related to the Design Centering problem studied by Vidigal and Director (1982) et al. We do not suppose that bdS, the boundary of S, is defined by convex or concave functions but only by C^2 functions and we assume that the constrained functions are known explicitly.

Generally speaking, in this kind of problem the subnorm is used, defined by a non-negative real valued function with sublinear, but for the sake of simplicity the discussion here will be confined to the case where K is the unit Euclidean ball B(0,1). Some new optimality conditions vill be proposed in this situation, that is, when Minkowski function coincides with the Euclidean norm $\|\cdot\|$.

Let S be the set defined by $\{x \in \mathbf{R}^n \mid f_i(x) \; f_i^0, i \in \Omega\}$ where $\Omega \triangleq \{1, \cdots, m\}$, $f_i \in C^2$ and f_i^0 are constant for $i \in \Omega$. For each $i \in \Omega$, we consider the set $D_i \triangleq \{y \in \mathbf{R}^n \mid f_i^0 \geq f_i^0$ and $f_j(y) \leq f_j^0, j \in \Omega \setminus \{i\}\}$ and we assume that S is bounded, simply connected and that $\nabla f_i(y) \neq 0$ for each $y \in D_i \cap \mathrm{bd}\, S, i \in \Omega$. Let $\hat{S} \subset \mathrm{int}\, S$ be closed and sufficiently close to S. Then the C_M-embedded problem can be formulated as follows:

$$\max_{x \in S} \min_{i \in \Omega} \min_{y \in D_i \cap S} \|x - y\|^2. \qquad (1)$$

It is equivalent to

$$\max_{x, r} r$$

$$\text{subject to } f_i(x + wr) \leq f_i^0, i \in \Omega, w \in \mathbf{B}(0,1),$$

Vidigal et al. (1982).

It is necessary to introduce an auxiliary function

$$\psi(x, \zeta, u) = \sum_{i \in \Omega} \omega_i(\zeta) \|x - u^i\|^2,$$

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wnere $\zeta \in I$, I being a bounded interval, $u \in X_{i\in\Omega} D_i$ and the function $\omega_i(\varepsilon)$ is defined by $(1/a_i)\prod_{j\in\Omega\setminus\{i\}}(\zeta - j)$, where $a_i = \prod_{j\in\Omega\setminus\{i\}}(i-j) = (-1)^{m-1}(i-1)!(m-1)!$, due to Demyanov (1968). The function ψ is continuously differentiable on $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^{n\times m}$ and the restriction of $\psi(x,\zeta,n)$ to $\hat{S}\times\Omega\times X_{i\in\Omega}D_i$ can be viewed as $\psi(x,\zeta,y) = \|x-y\|^2$ for $y \in D_\zeta$, $\zeta \in \Omega$, $x \in S$.

In the sequel we denote by $\varphi: \mathbf{R}^n \to \mathbf{R}$ the function defined for each $x \in S$ by $\varphi(x) \triangleq \min_{i\in\Omega} \min_{y\in D_i\cap S} \|x-y\|^2$ and we define to each $x \in \hat{S}$ the sets $\Omega(x) \triangleq \{i \in \Omega \mid \varphi(x) = \min_{y\in D_i\cap S} \|x-y\|^2\}$ and $Y(x) \triangleq \{y \in \mathrm{bd}\ S \mid \varphi(x) = \|x-y\|^2\}$. Finally we denote by $G(x;Q)$ the set $x-Q$ and by $N^+(x;Q)$ the cone $\{h \in \mathbf{R}^n \mid \langle u,h\rangle \geq 0, u \in G(x;Q)\}$. According to Demyanov et al. (1981, 1983) and Polak et al. (1979), we obtain the following characterization of φ.

Proposition 1. The function φ is quasidifferentiable, especially superdifferentiable in $\hat{S}$ in the sense of Demyanov and Rubinov and the superdifferential of φ at $x \in \hat{S}$ is $\partial\varphi(x) = \mathrm{co}2\,(x - y(x))$. Moreover φ is semi-smooth in the sense of Mifflin. □

From this proposition it follows that

$N^+(x;Y(x)) = \{h \in \mathbf{R}^n \mid \langle w,h\rangle \geq 0, w \in \partial\varphi(x)\}$.

By Demyanov (1981, 1983) and Polyakova (1981), a necessary condition for (1) is $0 \in \partial\varphi(x^*)$. This condition is equivalent to $0 \in \mathrm{co}(x^* - Y(x^*))$ or still to $\dim L_{C\partial\varphi(x^*)} \neq 0$, where CA means the conical hull of A and "$L_{(\cdot)}$" means the lineality space of $(\cdot)$, Stoer (1970) and Rockafellar (1970).

The following sufficient condition can be proved from Mifflin (1977), Lebourg (1975) and Polak (1979).

Theorem2. If $x \in S$ satisfies the condition $N^+(x;Y(x)) = \{0\}$ then x is a locally optimal solution to (1). In particular this sufficient condition holds when $Y(x) \cap \mathrm{int}\ \{CG(x;Q) + x\} \neq \emptyset$ for some $Q \subset Y(x)$. □

At this moment it must be mentioned that this sufficient condition is still true for more general cases. Note that if $\varphi'(x;h) = \varphi^\circ(x;h)$, $h \in \mathbf{R}^n$, then the proof for the theorem above can be simplified.

In order to discuss further optimality conditions, we begin by classifying the points of $Y(x)$ and afterwards the points $x \in \mathrm{int}\ S$.

A SP(y) B means that sets A and B can be separated by some hyperplane $H(y)$ at the point y. A SS(y) B means that A and B are in the same closed half space determined by a supporting hyperplane $H(y)$ of A and B at $y \in A \cap B$. $H(y; \nabla f_i(y), >)$ and $H(y; \nabla f_i(y), <)$ denote positive and negative open-half space, respectively, determined by $\nabla f_i(y)$ at y. Define

$\check{P}(x) \triangleq \{y \in Y(x) \mid \exists\ \varepsilon(y) > 0 : B(x, \phi(x)^{1/2})\ \mathrm{SP}(y) U_{\varepsilon(y)}(y) \cap \mathrm{bd}\ S\}$,

$\hat{P}(x) \triangleq \{y \in Y(x) \mid \exists \varepsilon(y) > 0 : B(x, \varphi(x)^{1/2})\ SS(y) U_{\varepsilon(y)}(y) \cap \mathrm{bd}\ S\}$,

$\tilde{P}(x) \triangleq \{y \in Y(x) \mid \forall \varepsilon > 0 : U_\varepsilon(y) \cap \mathrm{bd}\ S \cap H(y; \nabla f_i(y), >) \neq \phi$

and $U_\varepsilon(y) \cap \mathrm{bd}\ S \cap H(y; \nabla f_i(y), <) \neq \emptyset$, for some $i \in \Omega(x)$

such that

$$\|y - x\|^2 = \varphi(x),$$
$$f_i(y) = f_i^\circ,$$
$$f_j(y) \leqslant f_j^\circ, j \in \Omega \setminus \{i\}\},$$

and

$\overline{P}(x) = \{y \in Y(x) \mid \forall \varepsilon > 0 : U_\varepsilon(y) \cap \mathrm{bd} S \cap B(x, \varphi(x)^{1/2}) \setminus \{y\} \neq \emptyset$

and $B(x, \varphi(x)^{1/2})\ \underline{SS(y)}\ U_\varepsilon(y) \cap \mathrm{bd}\ S\}$.

In other words, $y \in Y(x)$ is said to be in $\check{P}(x)$ if there exist a neighborhood $U_{\varepsilon(y)}(y)$ of y and a hyperplane H passing through y such that

$B(x, \varphi(x)^{1/2})$ and $U_{\varepsilon(y)}(y) \cap \mathrm{bd}\ S$ can be separated by H. (2)

For example, if f_i is concave and $y \in D_i \cap S$, then $y \in \check{P}(x)$. In a similar way, y is said to be in $\hat{P}(x)$ if condition (2) is replaced by $B(x, \varphi(x)^{1/2})$ and $U_{\varepsilon(y)}(y) \cap \mathrm{bd}\ S$ are in the same closed half space determined by H. For example, if f_i is convex and $y \in D_i \cap S$, then $y \in \hat{P}(x)$. A point x is said to be a $\vee$-point ($\wedge$-point respectively) if $x \in \mathrm{int}\ S$ and $\check{P}(x) = Y(x)$ ($\hat{P}(x) = Y(x)$ respectively). Finally a $\vee$ or $\wedge$-point x is called strict if

$B(x, \varphi(x)^{1/2}) \cap U_{\varepsilon(y)}(y) \cap \mathrm{bd}\ S = \{y\}$. (3)

Theorem 3. Suppose x is a strict $\vee$-point. Then x is a locally optimal solution to (1) if and only if $N^+(x; Y(x)) = \{0\}$. □

Concerning the set P(x) and the $\wedge$-points, we the next results.

Proposition 4. If each $y \in \hat{P}(x)$ satisfies (3), then the condition dim $L_{C(x-P(x))} \neq 0$ is a sufficient one for x to be a locally optimal solution to (1). □

Theorem 5. Suppose x is a $\wedge$-point. Then x is a locally optimal solution to (1) if and only if $\dim N^+(x; Y(x)) \neq n$. □

In the general situation the main trouble is that it is possible that although the norm boby is not able to expand in a straight line, it can be able to expand in a curvilinear path. The general optimality condition can be described as follows.

Theorem 6. A point $x \in \mathrm{int}\ S$ is a strictly locally optimal solution to (1) if and only if there exists a neighborhood U(x) such that

$$\bigcap_{y_i \in Y(x)} [D_i + \|x - y_i\| B(0,1)]^c \cap U(x) = \emptyset. \quad (4)$$

If each function f_i is approximated by its second-order Taylor expansion, then (4) becomes more tractable and can be written as

$$\bigcap_{y_i \in Y(x)} \{z \mid 2\nabla f_i(y_i)^T (z - x) + (x - z)^T H_{y_i} (z - x) < 0\} \cap U(x) = \emptyset. \ \square$$

Th.3 and Th.5 have been found out exactly for the c_M-embedded problem (1). They can be used to verify if a point found out by any technique is a solution to (1) associated with the non-convex regions mentioned in the corresponding theorems.

Finally it remains to analyse the local behavior of the boundary at a point in order to identify if $y \in \hat{P}(x)$ or $\check{P}(x)$. This can be done with the aid of quadratic approximations as descrided in the last proposition.

Proposition 7. Suppose $y \in Y(x)$. If $y \in \check{P}(x)$ $(\widehat{P}(x))$, then there exists a neighborhood $U(y)$ such that $\triangle(y_T') \leqslant 0$ $(\geqslant 0)$, for each $y_T' \in Tf(y) \cap U(y)$, where $\triangle(y_T') \triangleq (y_T' - y)^T H_y (y_T' - y) / (2 \|\nabla f(y)\|^2)$ and $Tf(y)$ denotes the tangent plane to f at y. f at y. Conversely, if there exists $U(y)$ such that $\triangle(y_T') > 0$ (< 0) for each $y_T' \in Tf(y) \cap U(y)$ then $y \in \widehat{P}(x)$ $(\check{P}(x))$. □

Some corresponding algorithms are omitted here.

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