Journal of Mathematical Research and Exposition
vol.9, No.1, Feb., 1989

Vector-valued Pseudomeasures

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I Introduction

In this section, we list some definitions and results which are needed in what follows. The proofs of these facts are similar to those in scalar-valued functions. Hence we omit the details.

1. Let X be a Banach algebra, G a locally compact Abelian group, Σ collection of the Borel sets on G, λ a Haar measure on G, $\psi(G, X)$ the class of X-valued simple functions on G; i.e., functions of the form $\sum_{i=1}^{n} x_i \chi_{E_i}$ with $\{x_i\} \subset X$ and $\{E_i\}_1^n \subset \Sigma$, where χ_{E_i} is the characteristic function on the Borel set E_i $E_i \cap E_j = \phi$ when $i \neq j$ Then

$$(f_n * g_m)(t) = \int_G f_n(\tau) g_m(t\tau^{-1}) d\lambda(\tau)$$

is contained in $\psi(G, X)$ whenever $_n = \sum_{i=1}^{n} x_{1i} \chi_{E_{1i}}$, $g_m = \sum_{j=1}^{m} x_{2j} \chi_{E_{2j}}$. Moreover, $\psi(G, X) \subset L^1(G, X)$ where $L^1(G, X)$ denotes the class of all X-valued Bochner integrable functions on G, and the norm of an element in $L^1(G, X)$ is defined as $\|f\|_{L^1} = \int_G \|f(t)\|_x d\lambda(t)$. Obviously, the mapping

$$T: \psi(G, X) \times \psi(G, X) \to L^1(G, X) (f_n\ \ g_m) \mapsto f_n * g_m$$

is a bounded bilinear operator, where $\psi(G, X) \times \psi(G, X)$ denotes direct product.

By the extension for a bounded bilinear operator, there exists an unique extension of T and $\|T(f, g)\|_{L^1} \leqslant \|f\|_{L^1} \|g\|_{L^1}$ for all $f, g \in L^1(G, X)$. Owing to above fact we shall write $T(f, g)$ by notation $f*g$ and refer $f*g$ as convolutions of f and g.

It is possible to show that above definition for the convolution is coinci dent to usual sense of $f*g$.

2. By the definition of convolution in $L^1(G, X)$ we have that $L^1(G, X)$ is a Banach algebra with convolution as multiplication.

Let $\gamma \in \widehat{G}$ the dual group of G, $\|\cdot\|_{L^\infty}$ be the essential supremum norm.

* *Received* *ept*.20, 1986.

Then following formula is true

$$(f*g)^{\wedge}(\gamma)=\hat{f}(\gamma)\hat{g}(\gamma);\ \|(f*g)^{\wedge}\|_{L^\infty}\leqslant\|f\|_{L}\|g\|_{L^1};\ \|f*g\|_{L^1}\leqslant\|f\|_{L^1}\|g\|_{L^1}$$

for all $f, g\in L^1(G, X)$.

Since $\hat{f}=\hat{g}$ implies $f=g$ for all $f, g\in L^1(G, X)$ (see [1]), $L^1(G, X)$ is a semisimple Banach algebra.

3. **Theorem** Let X^* be the dual space of X, put $A(G, X^*)=\{\hat{f}(t)\mid f\in L^1(\hat{G}, X^*)\}$. Then $A(G, X^*)$ is a Banach algebra under pointwise operation and with the norm $\|\hat{f}\|_{A(X^*)}=\|f\|_{L^1}$

Proof Apply 1 and 2.

Ⅱ. The vector-valued pseudomeasures

1. **Definition** Collection of all continuous linear functionals on $A(G, X^*)$ is called vector-valued pseudomeasures and denoted by $P(G, A^*)$. We shall take notation $\|\cdot\|_P$ as the norm in the Banach space $P(G, A^*)$.

2. **Theorem** Let G be a locally compact Abelian group. Then

(i) $m(G, X)\hookrightarrow m(G, X^{**})\subset P(G, A^*)$. Here $m(G, X)$ denotes the class of all X-valued bounded measures on G; i.e., X-valued bounded semivariation measures on G, and the notation $\hookrightarrow$ denotes isometric embedding.

(ii) $M(G, X)\hookrightarrow M(G, X^{**})\hookrightarrow P(G, A^*)$. Here $M(G, X)$ denotes the class of all X-valued bounded variation measures on G.

Proof (i) Let $F\in m(G, X^{**})$. We have

$$\langle\hat{f}, F\rangle=\langle f, \hat{F}\rangle=\int_G f(r)\hat{F}(r)d\eta(r),$$

which is valid for each $\hat{f}\in A(G, X^*)$ (see [1]), where η is the Haar measure on G. Since the following inequalities

$$|\langle\hat{f}, F\rangle|\leqslant\|\hat{F}\|_{L^\infty}\|f\|_{L^\infty}=\|\hat{F}\|_{L^\infty}\|f\|_{A(X^*)}$$

is obviously true, F has been defined a bounded linear functional on $A(G, X^*)$ as equality $\langle\hat{f}, F\rangle=\int\hat{f}(t)dF(t)$, then $F\in P(G, A^*)$, $m(G, X^{**})\subset P(G, A^*)$. The inclusion relation $m(G, X)\subset m(G, X^{**})$ is obvious.

(ii) The conclusion is apparent from discussing for (i) and the inclu-sion relation $M(G, X^{**})\hookrightarrow m(G, X^{**})$ (see [1])

Ⅲ On the Fourier transform for elements in $P(G, A^*)$

1. **Lemma** $P(G, A^*)$, as in definition Ⅱ.1, is isometrically isomorphic to $L^1(\hat{G}, X^*)^*$, the conjugate space of $L^1(\hat{G}, X^*)$.

Proof Let $\sigma\in P(G, A^*)$. We define the mapping

$$L_\sigma:\ L^1(\hat{G}, X^*)\to C,\ f\mapsto\langle\hat{f}, \sigma\rangle.$$

Here $\langle\hat{f}, \sigma\rangle$ will denote the pairing between elements of $A(G, X^*)$ and

$P(G, A^*)$, C is the set of all complex numbers. We first note that from definition in Ⅱ.1 L_σ is well defined. Because $|L_\sigma(f)| = |\langle \hat{f}, \sigma\rangle| \leqslant \|\sigma\|_P \|\hat{f}\|_{A(X^*)} = \|\sigma\|_P \|f\|_{L^1}$, therefore L_σ is a bounded linear functional on $L^1(\hat{G}, X^*)$ whose operator norm $\|L_\sigma\|$ is not greater than $\|\sigma\|_P$.

Suppose, conversely, that $L_\sigma \in L^1(\hat{G}, X^*)^*$. Then $\langle f, L_\sigma\rangle$ is well defined when f belongs to $L^1(\hat{G}, X^*)$, since

$$|\langle f, L_\sigma\rangle| \leqslant \|L_\sigma\| \, \|f\|_{L^1} = \|L_\sigma\| \, \|\hat{f}\|_{A(X^*)},$$

thus $\langle f, L_\sigma\rangle$ has been defined a bounded linear functional on $A(G, X^*)$; ∴, there exists a $\sigma \in P(G, A^*)$ such that $\langle \hat{f}, \sigma\rangle = \langle f, L_\sigma\rangle$ and $\|\sigma\|_P \leqslant \|L_\sigma\|$. By preceding inequality $\|L_\sigma\| \leqslant \|\sigma\|_P$, we conclude from eqqality $\langle \hat{f}, \sigma\rangle = \langle f, L_\sigma\rangle$ that σ is isometrically corresponding to L_σ where $\sigma \in P(G, A^*)$, $L_\sigma \in L^1(\hat{G}, X^*)^*$.

2. **Corollary** If X^{**} possesses the Radon—Nikodym property, then $P(G, A^*)$ is isometrically isomorphic to $L^\infty(\hat{G}, X^{**})$. In this time we shall write $P(G, X^{**})$ as $P(G, A^*)$, where $L^\infty(\hat{G}, X^{**})$ is the class of X^{**}-valued essentially bounded measurable functions on $\hat{G}$ Certainly, $P(G, A^*)$ is isometrically isomorphic to $L^\infty(\hat{G}, X)$ when X is self-conjugate.

Proof We can see from [2], that $L^1(\hat{G}, X^*)^* = L^\infty(\hat{G}, X^{**})$. Thus, the conclusion in Corollary 2 is true.

3. **Definition** In the view point of Lemma Ⅲ. 1, let $\sigma \in P(G, A^*)$. Then there exists a unique L_σ such that $\langle \hat{f}, \sigma\rangle = \langle f, L_\sigma\rangle$ for any $f \in L^1(\hat{G}, X^*)$. L_σ will be known as the Fourier transform of σ, and denoted by $\hat{\sigma}$. From this definition it is immediately seen that the equality $\langle \hat{f}, \sigma\rangle = \langle f, \hat{\sigma}\rangle$ holds for any vector-valued pseudomeasure $\sigma \in P(G, A^*)$. Set $P(G, A^*)^\wedge = \{\hat{\sigma} | \sigma \in P(G, A^*)\}$, then

$$P(G, A^*)^\wedge = L^1(G, X^*)^*.$$

4. **Corollary** If X^{**} possesses the Radon-Nikodym property, then equality $P(G, X^{**})^\wedge = L^\infty(\hat{G}, X^{**})$ holds.

Proof Apply the conclusions of lemma Ⅲ. 2 and definition Ⅲ.3.

5. **Theorem** (i) The inclusion relations

$$L^1(G, X) \subset L^1(G, X^{**}) \subset P(G, A^*)$$

hold. Furthermore, when $f \in L^1(G, X^*)$, the Fourier transform of vector-valued function f coincides to the Fourier transform of f as a vector-valued pseudomeasure.

(ii) There exist properties of $m(G, X^{**})$ and $M(G, X^{**})$ similar to (i).

Proof Let us take arbitrary element F in $m(G, X^{**})$. We see, from [1], that $\langle f, \hat{F}\rangle = \langle \hat{f}, F\rangle$ for all $f \in L^1(\hat{G}, X^*)$. Furthermore by the definition of the Fourier transform for vector-valued pseudomeasure and the uniqueness of representation for functional, it is evident that the Fourier transform of vectorɔunded semivariation measure F coincides to the Fourier transform of F as a vector-valued pseuaomeasure.

For arbitrary $f \in L^1(G, X^{**})$, the mapping

$$F:\quad (G, \Sigma) \to X^{**},\quad E \mapsto \int_E f d\lambda$$

defines an element in $M(G, X^{**})$ and $\hat{F}(r) = \hat{f}(r)$ for any $r \in \hat{G}$, thus when we note that $M(G, X^{**}) \subset m(G, X^{**})$, the conclusion in theorem will be obtained.

Ⅳ Convolution of two vector-valued pseudomeusures

1. **Definition** Suppose that $L^1(\hat{G}, X^*)^*$ is a Banuch algebra Then, for all $\sigma_1, \sigma_2 \in P(G, A^*)$, $\hat{\sigma}_1 \cdot \hat{\sigma}_2 \in L^1(\hat{G}, X^*)^*$ as $\hat{\sigma}_1, \hat{\sigma}_2 \in L^1(\hat{G}, X^*)^*$ where denotes multiplication in $L^1(\hat{G}, X^*)^*$. Recalling the preceding definition of the Fourier transform for σ (see Ⅲ.3) it is evident that there exists an unique $\sigma \in P(G, A^*)$ such that $\hat{\sigma} = \hat{\sigma}_1 \cdot \hat{\sigma}_2$. We define $\sigma = \sigma_1 * \sigma_2$ and σ will be known as convolution of σ_1 and σ_2.

This definition is meaningful according to preceding disc sion. Obviously $(\sigma_1 * \sigma_2)^\wedge = \hat{\sigma}_1 \cdot \hat{\sigma}_2$ for all $\sigma_1, \sigma_2 \in P(G, A^*)$, and $\|\sigma_1 * \sigma_2\|_P = \|(\sigma_1 * \sigma_2)^\wedge\| = \|\hat{\sigma}_1 \cdot \hat{\sigma}_2\| < \|\hat{\sigma}_1\| \, \|\hat{\sigma}_2\| = \|\sigma_1\|_P \|\sigma_2\|_P$ where $\|\cdot\|$ denotes the norm of the Banach space $L^1(\hat{G}, X^*)^*$ (see Ⅲ.1 and Ⅲ.3). Furthermore, if $\hat{\sigma}_1 = \hat{\sigma}_2$, then $\langle \hat{f}, \sigma_1 - \sigma_2 \rangle = \langle f, \hat{\sigma}_1 - \hat{\sigma}_2 \rangle = 0$ for all $f \in L^1(\hat{G}, X^*)$. Consequently, $\sigma_1 - \sigma_2 = 0$, thus $\sigma_1 = \sigma_2$

We summarize the above discussion into the following theorem.

2. **Theorem** If $L^1(\hat{G}, X^*)^*$ is a Banach algebra, then $P(G, A^*)$ is a semisimple Banach algebra.

3. **Example** If X^{**} has the Radon—Nikodym property, then $L^1(\hat{G}, X^*)^* = L^\infty(\hat{G}, X^{**})$[2], and the latter is a Banach algebra under pointwise operations and with the essential supremum norm of space $L^\infty(\hat{G}, X^{**})$. Consequently, $P(G, X^{**})$ is a semisimple Banach algebra.

Ⅴ Remarks

1. Suppose X^{**} do not possess the Radon—Nikodym property. Since $L^\infty(\hat{G}, X^{**}) \subsetneq L^1(\hat{G}, X^*)^*$, thus $L^\infty(\hat{G}, X^{**})$ is a proper subste of $P(G, A^*)^\wedge$ (see [2])

2. Even though we X is self-conjugate and possesses the Radon—Nikodym property. equality $m(G, X^{**}) = P(G, X^{**})$ also cannot hold, when G is an infinite locally compact group.

In general, there exists a $\varphi \in C_C(\hat{G})$ such that $\varphi \notin m(G)^\wedge$, where $C_C(\hat{G})$ denotes the set of complex-valued continuous functions on $\hat{G}$ with compact support and $m(G)^\wedge$ is the set of the Fourier transforms for all bounded variation measure on G (see [2]). Thus $x^{**}\varphi \in C_C(\hat{G}, X^{**})$ for all $x^{**} \in X^{**}$. We can prove $x^{**}\varphi \notin m(G, x^{**})^{\wedge}$. Now we assume that $x^{**}\varphi$ belongs to $m(G, X^{**})^{\wedge}$, then there exists an unique $F \in m(G\ X^{**})$ such that $\hat{F}(r) = x^{**}\varphi(r)$. When we choose $x^{***} \in X^{***}$ so as to satisfy $\langle x^{**}, x^{***}\rangle = 1$, it is easily seen that $\langle \hat{F}(r), x^{***}\rangle = \langle F, x^{***}\rangle^{\wedge}(r)$ (see [1]). But since $\langle \hat{F}(r), x^{***}\rangle = \varphi(r)$ and $\langle F, x^{***}\rangle$ is a bounded variation scalar measure $\varphi \in m(G)^{\wedge}$. This contradicts the assumption $\varphi \notin m(G)^{\wedge}$.

From preceding proof we see that $\{x^{**}\varphi \mid x^{**} \in X^{**}, \varphi$ is a fixed element in $C_C(\hat{G})\} \subset C_C(\hat{G}, X^{**}) \subset L^{\infty}(\hat{G}, X^{**}) = P(G, X^{**})^{\wedge}$, and same set cannot belongs to $m(G, X^{**})^{\wedge}$, therefore $m(G, X^{**}) \subsetneqq P(G, X^{**})$.

3. It should be noted that the norm of elements in $m(G, X^{**})$ and $M(G, X^{**})$ as a vector-valued pseudomeasure is, in general, less than its norm as a vector-valued measure, for example, $\|\mu\|_P = \|\hat{\mu}\|_{L^\infty} < \|\mu\|$ when $\mu \in M(G, X^{**})$.

Ⅵ On vector-valued pseudomeasures in $L^2(G, X)$

1. **Lemma** Let the set of functions in $A(G, X)$ with compact support be denoted by $A_C(G, X)$. Then $A_C(G, X)$ is dense in $A(G, X)$ and $L^2(G, X)$, where $L^2(G, X)$ is the class of X-valued square integrable functions on G.

Proof Since $\Psi(\hat{G}, X)$ is dense in $L^1(\hat{G}, X)$, thus

$$A_{\psi}(G, X) = \{\hat{f}(t) \mid f \in \Psi(\hat{G}, X)\}$$

is dense in $A(G, X)$. Suppose $A_C(G)$, $A(G)$ and $A_{\psi}(G)$ equal to $A_C(G, C)$, $A(G, C)$ and $A_{\psi}(G, C)$ respectively, where C is the set of all complex numbers (see [3]). Since $A_C(G)$ is dense in $A_{\psi}(G)$ because $A_C(G)$ is dense in $A(G)$, then for any

$$\hat{f}_n = \sum_{i=1}^{n} x_i \hat{\chi}_{E_i} \in A_{\psi}(G, X)$$

there is some $\hat{g}_n = \sum_{i=1}^{n} x_i \hat{h}_i$, such that $\|\hat{f}_n - \hat{g}_n\|_{A(X)} < \varepsilon$ where ε is a small positive scalar, $\hat{h}_i \in A_C(G)$ and

$$\|\hat{\chi}_{E_i} - \hat{h}_i\|_{A(C)} < \varepsilon / n \|x_i\|_X.$$

Clearly, $\hat{g}_n$ belongs to $A_C(G, X)$. Thus for any $\varepsilon > 0$ and $\hat{f} \in A(G, X)$, there is some $\hat{f}_\varepsilon \in A_{\psi}(G, X)$ and some $\hat{g}_\varepsilon \in A_C(G, X)$ such that.

$$\|\hat{f} - \hat{f}_\varepsilon\|_{A(X)} < \varepsilon, \quad \|\hat{g} - \hat{g}_\varepsilon\|_{A(X)} < \varepsilon.$$

Via simple computation we have

$$\|\hat{f} - \hat{g}_\varepsilon\|_{A(X)} < 2\varepsilon.$$

From previous estimation, we conclude that $A_C(G, X)$ is dense in $A(G, X)$.

The proof of denseness for $A_C(G, X)$ in $L^2(G, X)$ can be obtained by the similar procedure, we only note the denseness of $\Psi(G, X)$ in $L^2(G, X)$ and

$A_C(G)$ in $L^2(G)$. Here $L^2(G)$ denotes the class of complex-valued square integrable functions on G.

2 . The Plancherel formula in $L^2(G, H)$. Using the plancherel theorem in $L^2(G,H)$ we can define the Fourier transformation $\hat{f}$ of an element f in $L^2(G, H)$ where H is the Hilbert space (see [1]). In particular, $\hat{f}$ will be called the Fourier-Plancherel transformation of f when $f \in L^2(G, H)$.

For any $f \in L^2(\hat{G}, H)$ we shall define

$$\hat{f}(t) = \int (t, r) f(r) d\eta(r)$$

as the Fourier trunsform of f By these definitions we can easily conclude that formula

$$(\hat{f})^\wedge = (\check{f})^\vee = f$$

holds for all $f \in L^2(G, H)$. Here $\check{f}$ is the inversion of f by the Plancherel theorem in $L^2(G, H)$. Using this equality and the Parsevel formula in $L^2(G, H)$ (see [1]), we can obtain the following the Plancherel formula

$$\int_G \langle \hat{f}(t), g(t) \rangle \, d\lambda(t) = \int \langle \hat{f}(t), (\hat{g})^\wedge(t) \rangle_H d\lambda(t) = \int \langle f(r), \hat{g}(r) \rangle_H d\eta(r)$$

where $\langle \cdot, X \rangle_H$ denotes the inner-product in H, and $f \in L^2(\hat{G}, H)$, $g \in L^2(G, H)$.

3 . **Definition** We shall say that $\sigma \in P(G, A^*)$ belongs to $L^2(G, X)$ if there exists a $g \in L^2(G, X)$ such that $\langle \hat{f}, \sigma \rangle = \langle \hat{f}, g \rangle$ for all $\hat{f} \in A_C(G, X^*)$ Since $A_C(G, X^*)$ is dense in $L^2(G, X^*)$ (see Ⅵ.1), g is uniquely determined

4 . **Theorem** Let X be the Hilbert space H If σ is vector-valued pseudomeasure in $L^2(G, H)$ (see Ⅵ.3), then the Fourier transform of σ as a H-valued pseudomeasure and the Fourier-Plancherel transformation of σ as an element of $L^2(G, H)$ agree .

Proof Since H is self-conjugate and has the Radon-Nikodym property [3]

$$P(G, H) = A(G, H^*)^* = A(G, H)^*,$$

$$P(G, H)^\wedge = L^\infty(G, H) = L^1(\hat{G}, H^*)^* = L^1(\hat{G}, H)^*,$$

(see Ⅱ.1 and Ⅲ.2), suppose $\sigma \in P(G, H) \cap L^2(G, H)$, by definition in Ⅵ.3 we co conclude that there is an unique $g \in L^2(G, H)$ such that $\langle \hat{f}, \sigma \rangle = \langle \hat{f}, g \rangle$ for all $\hat{f} \in A_C(G, H)$. An easy computation using property of the Fourier transform for v vector-valued pseudomeasure and the Plancherel formula (see Ⅵ.2) shows for each $\hat{f} \in A_C(G, H)$ that

$$\langle f, \hat{\sigma} \rangle = \langle \hat{f}, \sigma \rangle = \langle \hat{f}, g \rangle = \langle f, \hat{g} \rangle .$$

Appealing to the denseness $A_C(G, H)$ in $A(G, H)$ (see Ⅵ.1) we conclude that above equality holds for all $f \in L^1(\hat{G}, H)$. Therefore $\hat{\sigma} = \hat{g}$.

5 . **Theorem** If $\sigma \in P(G, H)$, then $\sigma \in L^2(G, H)$ if and only if $\hat{\sigma} \in L^2(\hat{G}, H) \cap L^\infty(\hat{G}, H)$

Proof Necessity is immediately apparent from theorem Ⅵ.4. Now we prove that the condition is sufficient .

If $\hat{\sigma} \in L^2(\hat{G}, H)$, then the mapping

$$T: L^2(G, H) \to C, \quad f \mapsto \langle \hat{f}, \hat{\sigma} \rangle$$

defines a continuous linear functional on $L^2(G, H)$ because

$$|Tf| = |\langle \hat{f}, \hat{\sigma} \rangle| \leq \|\hat{f}\|_{L^2} \|\hat{\sigma}\|_{L^2} = \|f\|_{L^2} \|\sigma\|_{L^2}.$$

Thus there is unique $g \in L^2(G, H)$ such that

$$\langle \hat{f}, \hat{\sigma} \rangle = Tf = \langle f, g \rangle$$

for all $f \in L^2(G, H)$, using the Plancherel formula in $L^2(G, H)$ (see Ⅵ.2) and the property of vector-valued pseudomeasure we conclude that for all $f \in L^2(G, H)$ the equality $\langle f, \sigma \rangle = \langle (\hat{f})^\wedge, \sigma \rangle = \langle \hat{f}, \hat{\sigma} \rangle = \langle f, g \rangle$ holds. Since $A_C(G, H)$ is dense in $L^2(G, H)$, obviously, $\langle f, \sigma \rangle = \langle f, g \rangle$ for all $f \in A_C(G, H)$ By definition in Ⅵ.3 we have $\sigma \in L^2(G, H)$.

The proof of theorem is complete.

References

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Theorem 3 Let $\{X_n\}$ be a strictly stationary φ-mixing sequence of random variables with $EX_1 - 0, EX_1^2 < \infty$ and $\sigma_n^2 = ES_n^2 \to \infty$ as $n \to \infty$. If $\sum_{n=1}^{\infty} \varphi^{1/2}(2^n) < \infty$, then, without changing the distribution of $\{S(t)\}$ we can redefine $\{S(t)\}$ on a richer probability space together with a standard Wiener process $\{W(t), t \geq 0\}$ such that

$$S(t) - W(\sigma_t^2) = o((\sigma_t^2 \log\log t)^{1/2}) \qquad \text{a.s.}$$

as $t \to \infty$. Where $S(t) = \sum_{i \leq t} X_i$ and $\sigma_t^2 = ES^2(t)$.