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A Moment Inequality and Its Applications*

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Marcikeiwicz-Zygmud inequality plays a very important rule in various proofs. However it is well-known that M-Z inequality foous on independent random variables. In this paper we establish a more useful inequality for φ-mixing sequence follows.

Theorem 1 Let $X_1, \cdots, X_n$ be a sequence of B-valyed random vectors and let $S_i = \sum_{j=1}^{i} X_j$. Suppose there exist integer $p \geq 1$ and constants $A > 0$ and $\varepsilon > 0$ such that $\varphi_n(p) + \max_{p \leq m \leq n} P(\|S_n - S_m\| \geq A) < \varepsilon$ where $\varphi_n(p) = \max_{1 \leq k \leq n-p} \varphi(F_1^k, F_{k+p}^n)$, $\varphi(F_1^k, F_{k+p}^n) = \sup_{c \in F_1^k, B \in F_{k+p}^n} |P(B|C) - P(B)|$, $F_a^b = \sigma(x_i, a \leq i \leq b)$. Then for every $q > 0$ we have

$$E \max_{1 \leq i \leq n} \|S_i\|^q \leq (1 - \varepsilon - 4^q \varepsilon)^{-1}((8A)^q + 2(4p)^q E \max_{1 \leq i \leq n} \|X_i\|^q)$$

provided $4^q \varepsilon < 1 - \varepsilon$.

Corollary Let $\{X_k, k \geq 1\}$ be a φ-mixing sequence of real-valued random variables and let $S_k(n) = \sum_{i=k+1}^{k+n} X_i (k \geq 0, n \geq 1)$. Suppose there exists a sequence of positive numbers $\{C_n\}$ such that for every $k \geq 0, n \geq 1, m \leq n$ $ES_k^2(m) \leq C_n$. Then for every $q \geq 2$, there is a constant K depending only on $\varphi(\cdot)$ and q such that for every $k \geq 0$, $n \geq 1$

$$E \max_{1 \leq i \leq n} |S_k(i)|^q \leq K(C_n^{q/2} + E \max_{k < i \leq k+n} |X_i|^q)$$

As applications we have

Theorem 2 Let $\{X_n\}$ be a φ-mixing sequence of identically distributed random variables. Then the following conditions are equivalent:

(i) $E|X_1|^{rt} < \infty$, $EX_1 = 0$;

(ii) $\sum_{n=1}^{\infty} n^{r-2} P(\max_{i \leq n} |S_i| > \varepsilon n^{1/t}) < \infty \qquad \varepsilon > 0$

(iii) $\sum_{n=1}^{\infty} n^{r-2} P(\sup_{k \geq n} |S_k|/k^{1/t} > \varepsilon) < \infty \qquad \varepsilon > 0$

where $r > 1$ and $1 \leq t < 2$. (to 23)

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that the condition is sufficient .

If $\hat{\sigma} \in L^2(\hat{G}, H)$, then the mapping

$$T: L^2(G, H) \to C, \quad f \mapsto \langle \hat{f}, \hat{\sigma} \rangle$$

defines a continuous linear functional on $L^2(G, H)$ because

$$|Tf| = |\langle \hat{f}, \hat{\sigma} \rangle| \leq \|\hat{f}\|_{L^2} \|\hat{\sigma}\|_{L^2} = \|f\|_{L^2} \|\sigma\|_{L^2}.$$

Thus there is unique $g \in L^2(G, H)$ such that

$$\langle \hat{f}, \hat{\sigma} \rangle = Tf = \langle f, g \rangle$$

for all $f \in L^2(G, H)$, using the Plancherel formula in $L^2(G, H)$ (see Ⅵ.2) and the property of vector-valued pseudomeasure we conclude that for all $f \in L^2(G, H)$ the equality $\langle f, \sigma \rangle = \langle (\hat{f})^\wedge, \sigma \rangle = \langle \hat{f}, \hat{\sigma} \rangle = \langle f, g \rangle$ holds. Since $A_C(G, H)$ is dense in $L^2(G, H)$, obviously, $\langle f, \sigma \rangle = \langle f, g \rangle$ for all $f \in A_C(G, H)$ By definition in Ⅵ.3 we have $\sigma \in L^2(G, H)$.

The proof of theorem is complete.

References

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(from 24)

Theorem 3 Let $\{X_n\}$ be a strictly stationary φ-mixing sequence of random variables with $EX_1 - 0, EX_1^2 < \infty$ and $\sigma_n^2 = ES_n^2 \to \infty$ as $n \to \infty$. If $\sum_{n=1}^{\infty} \varphi^{1/2}(2^n) < \infty$, then, without changing the distribution of $\{S(t)\}$ we can redefine $\{S(t)\}$ on a richer probability space together with a standard Wiener process $\{W(t), t \geq 0\}$ such that

$$S(t) - W(\sigma_t^2) = o((\sigma_t^2 \log\log t)^{1/2}) \quad \text{a.s.}$$

as $t \to \infty$. Where $S(t) = \sum_{i \leq t} X_i$ and $\sigma_t^2 = ES^2(t)$.