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On Deficient Values of Meromorphic Functions*

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Ⅰ. Let $f(z)$ be a meromorphic function of order $\rho\ (0<\rho<\infty)$ and $T(r, f)$ be the characteristic function. We use the standard notation of Nevanlinna theory [5]. If $\rho(r)$ is a proximate order relative to $T(r,f)$ Levln [1], then by using $\rho(r)$ Valiron [2] has introduced the following, for any complex number a, $\delta_{\rho(r)}(a)=1-\limsup\limits_{r\to\infty}\frac{N(r,a)}{r^{\rho(r)}}$ represents the $\rho(r)$-defect and if $\delta_{\rho(r)}(a)>0$, a is called $\rho(r)$-deficient value. Sarangi [3] proved that $N_{\rho(r)}(a)=\{a\,|\,\delta_{\rho(r)}(a)>0\quad a\in\bar{C}\}$ is countable and the following lemma.

Lemma (A): Let $f(z)$ be a non-integral order $\rho\ (0<\rho<\infty)$ then

$$K_{\rho(r)}(f)\geqslant 1-\rho \quad \text{for} \quad 0<\rho<1$$

$$K_{\rho(r)}(f)\geqslant\frac{(q+1-\rho)(\rho-q)}{c_1(q)} \quad \text{for } \rho>1 \text{ and } q=[\rho]$$

where $c_1(q)=0$ if $q=0$; $c_1(q)=2(q+1)[2+\log(q+1)]$ if $q>0$.

Ⅱ. In this paper we define $\delta_{\rho(r)}(a)=\liminf\limits_{r\to\infty}\frac{m(r,a)}{r^{\rho(r)}}$, find the bounds for $\frac{T(r,f')}{r^{\rho(r)}}$ and the other results.

Before that we state the following lemma whose proof follows easily from first fundamental theorem of Nevanlinna.

Lemma If $f(z)$ is a meromorphic function of order $\rho\ (0<\rho<\infty)$ then

$$\liminf_{r\to\infty}\frac{m(r,a)}{r^{\rho(r)}}\leqslant 1-\limsup_{r\to\infty}\frac{N(r,a)}{r^{\rho(r)}}$$

Theorem 1 If $f(z)$ is a meromorphic function of order $\rho\ (0<\rho<\infty)$ then

$$\sum_{a\neq\infty}\delta_{\rho(r)}(a)\leqslant\liminf_{r\to\infty}\frac{T(r,f')}{r^{\rho(r)}}\leqslant\limsup_{r\to\infty}\frac{T(r,f')}{r^{\rho(r)}}\leqslant 2-\Theta_{\rho(r)}(\infty)$$

where $\Theta_{\rho(r)}(\infty)=1-\limsup\limits_{r\to\infty}\frac{\bar{N}(r,f)}{r^{\rho(r)}}$

Proof The following inequalities are wellknown for any positive r

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$$N\left(r,\frac{1}{f'}\right)+\sum_{i=1}^{q} m(r,a_i)-S(r,f)\leqslant T(r,f') \qquad (1)$$

$$T(r,f')\leqslant T(r,f)+\bar{N}(r,f)+S(r,f) \qquad (2)$$

Now from (1): $\sum_{i=1}^{q} m(r,a_i)-S(r,f)\leqslant T(r,f')$. Hence

$$\sum_{i=1}^{q}\frac{m(r,a_i)}{r^{\rho(r)}}-O(1)\leqslant\frac{T(r,f')}{r^{\rho(r)}}.$$

Taking the limit inferior as $r\to\infty$, $\liminf_{r\to\infty}\sum_{i=1}^{q}\frac{m(r,a_i)}{r^{\rho(r)}}\leqslant\liminf_{r\to\infty}\frac{T(r,f')}{r^{\rho(r)}}$, i.e.

$$\sum_{i=1}^{q}\liminf_{r\to\infty}\frac{m(r,a_i)}{r^{\rho(r)}}\leqslant\liminf_{r\to\infty}\frac{T(r,f')}{r^{\rho(r)}}.$$

Hence from Lemma $\sum_{i=1}^{q}\delta_{\rho(r)}(a_i)\leqslant\liminf_{r\to\infty}\frac{T(r,f')}{r^{\rho(r)}}$. Hence

$$\sum_{a\neq\infty}\delta_{\rho(r)}(a)\leqslant\liminf_{r\to\infty}\frac{T(r,f')}{r^{\rho(r)}} \qquad (3)$$

Similarly from (2), $\frac{T(r,f')}{r^{\rho(r)}}\leqslant\frac{T(r,f)}{r^{\rho(r)}}+\frac{\bar{N}(r,f)}{r^{\rho(r)}}+O(1)$.

Taking limit superior as $r\to\infty$, $\limsup_{r\to\infty}\frac{T(r,f')}{r^{\rho(r)}}\leqslant\limsup_{r\to\infty}\frac{T(r,f)}{r^{\rho(r)}}+\limsup_{r\to\infty}\frac{\bar{N}(r,f)}{r^{\rho(r)}}$, i.e. $\limsup_{r\to\infty}\frac{T(r,f')}{r^{\rho(r)}}\leqslant 1+1-\Theta_{\rho(r)}(\infty)$, i.e.

$$\limsup_{r\to\infty}\frac{T(r,f')}{r^{\rho(r)}}\leqslant 2-\Theta_{\rho(r)}(\infty). \qquad (4)$$

Hence from (3) and (4) we have

$$\sum_{a\neq\infty}\delta_{\rho(r)}(a)\leqslant\liminf_{r\to\infty}\frac{T(r,f')}{r^{\rho(r)}}$$

$$\leqslant\limsup_{r\to\infty}\frac{T(r,f')}{r^{\rho(r)}}\leqslant 2-\Theta_{\rho(r)}(\infty). \blacksquare$$

Theorem 2 If $f(z)$ is a meromorphic function of order ρ $(0<\rho<\infty)$ tnen

$$\frac{1}{2-\Theta_{\rho(r)}(\infty)}\sum_{a\neq\infty}\delta_{\rho(r)}(a)\leqslant\delta(0,f')$$

Proof From the result $\sum_{i=1}^{q} m(r,a_i)-S(r,f)\leqslant m(r,1/f')$. We have

$$\sum_{i=1}^{q}\frac{m(r,a_i)}{r^{\rho(r)}}-O(1)\leqslant\frac{m(r,1/f')}{r^{\rho(r)}}$$

Taking limit inferior as $r\to\infty$

$$\sum_{i=1}^{q} \liminf_{r\to\infty} \frac{m(r,a_i)}{r^{\rho(r)}} \leqslant \liminf_{r\to\infty} \frac{m(r,1/f')}{r^{\rho(r)}}$$

$$\leqslant \liminf_{r\to\infty} \frac{m(r,1/f')}{T(r,f')} \limsup_{r\to\infty} \frac{T(r,f')}{r^{\rho(r)}} .$$

Hence from theorem 1 and lemma $\sum_{i=1}^{q} \delta_{\rho(r)}(a_i) \leqslant \delta(0,f')(2-\Theta_{\rho(r)}(\infty))$. Hence

$$\frac{1}{2-\Theta_{\rho(r)}(\infty)} \sum_{a\neq\infty} \delta_{\rho(r)}(a) \leqslant \delta(0,f').$$ ∎

Theorem 3 If $f(z)$ is a meromorphic function of order ρ $(0<\rho<\infty)$ such that $\sum_{a\neq\infty} \delta_{\rho(r)}(a) = 1$ and $\delta_{\rho(r)}(\infty) = 1$, then the order of the function is positive integer.

Proof We know that $\delta_{\rho(r)}(\infty) \leqslant \Theta_{\rho(r)}(\infty) \leqslant 1$. Since $\delta_{\rho(r)}(\infty) = 1$, we have

$$\Theta_{\rho(r)}(\infty) = 1 \tag{5}$$

Now from theorem 1 we have $1 \leqslant \liminf_{r\to\infty} \frac{T(r,f')}{r^{\rho(r)}} \leqslant \limsup_{r\to\infty} \frac{(r,f')}{r^{\rho(r)}} \leqslant 1$.

Hence

$$T(r,f') \sim r^{\rho(r)} \tag{6}$$

Since $\sum_{a\neq\infty} \delta_{\rho(r)}(a) = 1$, from theorem 2 and (5) we have $\sum_{a\neq\infty} \delta_{\rho(r)}(a) \leqslant \delta(0,f')$, i.e. $1 \leqslant \delta_{(0,f')} \leqslant 1$. Hence

$$\delta(0,f') = 1 \tag{7}$$

Also we know that $N(r,f') \leqslant 2N(r,f)$ and from (6)

$\limsup_{r\to\infty} \frac{N(r,f')}{T(r,f')} \leqslant 2 \limsup_{r\to\infty} \frac{N(r,f)}{r^{\rho(r)}} = 0$. Hence $\limsup_{r\to\infty} \frac{N(r,f')}{T(r,f')} = 0$.

Hence

$$\delta(\infty,f') = 1 \tag{8}$$

Now if $S(r)$ is a proximate order relative to $T(r,f')$, then from [3] $K_{S(r)}(f') \leqslant K(f') \leqslant 2-\delta(0,f')-\delta(\infty,f') = 0$. Hence $K_{S(r)}(f') = 0$, and from lemma (A) f' is of positive integral order. since order of f and f' are equal, the order of f is also a positive integer.

Ⅲ. In [4] Edrei and Fuchs proved the following theorem.

Theorem A Let $f(z)$ be a meromorphic function of order ρ. Let $K(f) = \limsup_{r\to\infty} \frac{N(r,0)+N(r,\infty)}{T(r,f)}$. Then

$$K(f)\begin{cases}\geqslant 1 & (0<\rho<1/2)\\ \geqslant \sin\pi\rho & (1/2\leqslant\rho<1)\end{cases}$$

In this section we prove that the above result is true for $K_{\rho(r)}(f)$, where

$$K_{\rho(r)}(f)=\limsup_{r\to\infty}\frac{N(r,0)+N(r,\infty)}{r^{\rho(r)}}.$$

Theorem 4 Let $f(z)$ be a meromorphic function of order ρ, where $0<\rho<1$. Then

$$K_{\rho(r)}(f)\begin{cases}\geqslant 1 & (0\leqslant\rho<1/2)\\ \sin\pi\rho & (1/2\leqslant\rho<1)\end{cases}$$

Proof It is sufficient to prove the theorem when $f(z)$ is of the form $f(z)=\prod_{n=1}^{\infty}(1+\frac{z}{a_n})/\prod_{n=1}^{\infty}(1+\frac{z}{b_n})$. We have from Hayman[5]

$$T(r,f)=\int_0^\infty N(t,0)P(t,r,\beta)dt+\int_0^\infty N(t,\infty)P(t,r,\pi-\beta)dt, \quad (9)$$

which is valid for all sufficiently larger and some $\beta=\beta(r)$ such that $0<\beta(r)<\pi$, $\log|f(re^{i\beta})|=0$. If $\theta=\theta(r)=\max\{\beta(r),\pi-\beta(r)\}$, then from (9), we have

$$\begin{aligned}T(r,f)&<\int_0^\infty\{N(t,0)+N(t,\infty)\}P(t,r,\theta)dt\\ &=\int_0^{r_0}\{N(t,0)+N(t,\infty)\}P(t,r,\theta)dt\\ &+\int_{r_0}^\infty\{N(t,0)+N(t,\infty)\}P(t,r,\theta)dt\\ &\leqslant\frac{O(1)}{r}+\int_{r_0}^\infty\{N(t,0)+N(t,\infty)\}P(t,r,\theta)dt\end{aligned}\quad (10)$$

Now, $K_{\rho(r)}(f)=\limsup_{r\to\infty}\frac{N(r,0)+N(r,\infty)}{r^{\rho(r)}}\leqslant 2$. Hence

$$N(r,0)+N(r,\infty)\leqslant(K_{\rho(r)}(f)+\varepsilon)r^{\rho(r)},\ \text{for}\ r\geqslant r_0.$$

Hence from (10) we have

$$\begin{aligned}T(r,f)&\leqslant\frac{O(1)}{r}+(K_{\rho(r)}+\varepsilon)\int_{r_0}^\infty t^{\rho(t)}P(t,r,\theta)dt.\\ &=\frac{O(1)}{r}+(K_{\rho(r)}+\varepsilon)\{\int_{r_0}^{r}t^{\rho(t)}P(t,r,\theta)dt\\ &+\int_{\delta_r}^{v_r}t^{\rho(t)}P(t,r,\theta)dt+\int_{v_r}^\infty t^{\rho(t)}P(t,r,\theta)dt\}\end{aligned}$$

where δ, δ, v are positive numbers such that $\delta<1$ and $v>1$ i.e.

$$T(r,f)\leqslant\frac{O(1)}{r}+(K_{\rho(r)}+\varepsilon)\{I_1+I_2+I_3\} \quad (11)$$

Now, since $0<r_0\leqslant t\leqslant\delta_r<r$ and $0<\theta<\pi$, $\sin\theta<1$ and $t^2+2tr\cos\theta+r^2\geqslant$

$(r-t)^2 \geqslant r^2(1-\delta)^2$. Hence

$$I_1 = \frac{1}{\pi}\int_{r_0}^{\delta_r} \frac{t^{\rho(t)} r\sin\theta}{t^2+2tr\cos\theta+r^2}\,dt = \frac{1}{\pi}\frac{1}{(1-\delta)^2 \cdot r}\int^{\delta r} t^{\rho(t)}dt$$

$$\sim \frac{1}{\pi(1-\delta)^2 r}\cdot\frac{(\delta_r)^{\rho(\delta_r)+1}}{\rho+1} = \frac{1}{\pi(1-\delta)^2 r}\frac{\delta^{\rho+1} r^{\rho(r)+1}}{\rho+1}$$

$$= \frac{\delta^{\rho+1} r^{\rho(r)}}{\pi(1-\delta)^2(\rho+1)} \tag{12}$$

Similarly,

$$I_2 = \frac{1}{\pi}\int_{\delta r}^{v_r} \frac{t^{\rho(t)} r\sin\theta}{t^2+2tr\cos\theta+r^2}\,dt$$

Putting $t=ru$, we have

$$I_2 = \frac{1}{\pi} r^{\rho(r)}\int_{\delta}^{v} \frac{u^{\rho}\sin\theta}{u^2+2u\cos\theta+1}\,du$$

$$= \frac{1}{\pi} r^{\rho(r)}\frac{\sin\theta\rho}{\sin\pi\rho} \text{ from contour integration} \tag{13}$$

Since $0<\theta<\pi$, $0<\sin\theta<1$, since $v>1$ and $t \geqslant v_r > r$, we have

$$t^2+2tr\cos\theta+r^2 \geqslant (t-r)^2 = t^2(1-\frac{r}{t})^2 > t^2(1-\frac{1}{v})^2$$

Hence

$$I_3 = \frac{1}{\pi}\int_{v_r}^{\infty} \frac{t^{\rho(t)} r\sin\theta}{t^2+2tr\cos\theta+r^2}\,dt \leqslant \frac{1}{\pi} r\int_{v_r}^{\infty}\frac{t^{\rho(t)}}{t^2(\frac{v-1}{v})^2}\,dt$$

$$= \frac{1}{\pi} r(\frac{v}{v-1})^2\int_{v_r}^{\infty} t^{\rho(t)-2}dt \sim \frac{1}{\pi}(\frac{v}{v-1})^2\cdot r\cdot\frac{(v_r)^{\rho(v_r)-1}}{1-\rho}$$

$$\sim \frac{1}{\pi}(\frac{v}{v-1})^2\cdot r\cdot\frac{v^{\rho-1} r^{\rho(r)-1}}{1-\rho} = \frac{1}{\pi}\frac{v^{\rho+1} r^{\rho(r)}}{(v-1)^2(1-\rho)} \tag{14}$$

Hence using the results (12), (13) and (14) in (11) we have

$$T(r,f) \leqslant \frac{O(1)}{r} + (K_{p(r)}+\varepsilon)\cdot r^{\rho(r)}\Big\{\frac{\delta^{\rho+1}}{\pi(1-\delta)^2(\rho+1)}$$

$$+\frac{\sin\theta\rho}{\sin\pi\rho} + \frac{v^{\rho+1}}{\pi(v-1)^2(1-\rho)}\Big\}$$

Dividing both sides by $r^{\rho(r)}$ and taking limit superior as $r\to\infty$ and letting $\delta\to 0$ and $v\to\infty$, gwe get $1 \leqslant K_{p(r)}\frac{\sin\theta\rho}{\sin\pi\rho}$. Hence $\sin\pi\rho = K_{p(r)}\max\limits_{0<\theta<\pi}\sin\theta\rho$. When $0<\rho<1/2$, $\max\limits_{0<\theta<\pi}\sin\theta\rho = \sin\pi\rho$ and when $1/2 \leqslant \rho < 1$, $\max\limits_{0<\theta<\pi}\sin\theta\rho = 1$. Hence

$$K_{\rho(r)}(f)\begin{cases}\geq 1 & (0<\rho<1/2)\\ \geq \sin\pi\rho & (1/2\leq\rho<1)\end{cases}.$$

Theorem 5 If $f(z)$ is a meromorphic function of order ρ where $0<\rho<1$ then

$$K_{\rho(r)}(f)=\lim_{r\to\infty}\sup\frac{N(r,0)+N(r,\infty)}{r^{\rho(r)}}\geq\frac{\sin\pi\rho}{\pi\rho}$$

Proof Follows from theorem 4.

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