Journal of Mathematical Research and Exposition
Vol.9, No.1, Feb., 1989

Extension of a Theorem of Carleson-Duren

Ren Fu-Yao and Huang Li-feng

(Department of Mathematics, Fudan University, Shanghai)

1. Introduction

A theorem of Carleson [1],[2] as generalized by Duren [3] characterizes those positive measure μ on the unit disc $U=\{z\in C:|z|<1\}$ for which the H^p norm domiates the $L^q(\mu)$ norm of elements of H^p. Later on, Hasting [5] proved an analogous results with H^p replaced by A^p, the Bergman space of fuctions f which $\int_0^1\int_0^{2\pi}|f(re^{i\theta})|^p r\,dr\,d\theta<\infty$. Actually, his result is more general in that it applies to positive measure and positive n-subharmonic functions on the unit polydisc U^n in C^n, the purpose of this article is to generalize the theorems of Duren and Hastings.

2. Extension of the theorem of Duren

Theorem 1 Let μ be a finite, positive measure on U, and suppose that the function $\phi(t):[0,\infty)\to R$ satisfies the following conditions:

(i) $\phi(0)=0$, $\phi(t)>0$, $t>0$,

(ii) ϕ is increasing and $\lim\frac{\phi(t)}{t}=\infty$ or finite,

(iii) ϕ' exists and is increasing in $(0,\infty)$,

(iv) $\lim\limits_{t\to 0}\phi(ct^{-1})\phi(t^2)=0$,

(v) there exists a constant $B>0$ such that

$$\sup_{t>0}\frac{t\phi'(t)\phi(ct^{-1})}{\phi(c)}=B$$

for all $c>0$. Then in order that there exists a constant $C>0$ depending only on ϕ such that

$$\phi^{-1}\{\int_U\phi(|f(z)|^p)\,d\mu(z)<C\|f\|_p^p \tag{1}$$

for all $f\in H^p$, $0<p<\infty$, it is necessary and sufficient that there is a positive constant A depending only on ϕ such that

* Received Oct.27, 1986.

$$\mu(S_h) \leq \phi(Ah) \tag{2}$$

for every set S_h of the form

$$S_h = \{z = re^{i\theta} : 1 - h \leq r < 1, \theta_0 \leq \theta \leq \theta_0 + h\} . \tag{3}$$

We need the following lemma which is obtained by elementary calculus.

Lemma Suppose that function $\phi(t): [0, \infty) \rightarrow \mathbf{R}$ satisfies the following conditions:

(i) $\phi(0) = 0, \phi(t) > 0,\ t > 0,$

(ii) ϕ is increasing and

(iii) ϕ' exists and is increasing in $(0, \infty)$.

Then the following properties are true:

$$\phi(ct) = c\phi(t), \text{ for } c \geq 1, \tag{4}$$

$$\phi(t)/t \text{ is increasing in } (0, \infty), \tag{5}$$

$$\phi(t_1) + \phi(t_2) \leq \phi(t_1 + t_2) . \tag{6}$$

for all $t_1, t_2 \in [0, \infty)$.

Proof of necessity Suppose that (1) holds with $p, 0 < p < \infty$, it is easy to see that

$$\mu(S_h) = \phi(c \| f \|_p^p) / \phi(\min_{z \in S_h} | f(z) |^p) \tag{7}$$

for all $f \in H^p$ and for every set S_h of the form (3).

Let $z_0 = \rho e^{i\alpha}$, and let $\rho = 1 - h$, and consider the H^p function $f(z) = [5h^2(1-z_0 z)^{-2}]^{1/p}$, whose norm is $\| f \|_p^p = 5h^2(1 - \rho^2)^{-1} < 5h$.

A geometric argument [4, p. 157] shows that $| f(z) |^p \geq 1$ in S_h. Therefore, by (7), $\mu(S_h) = \phi(5ch)/\phi(1)$ and (2) holds with $A = 5c/\phi(1)$ for $\phi(1) < 1$ and $A = 5c$ for $\phi(1) \geq 1$.

Proof of sufficiency Suppose that (2) holds for every set S_h of the fo form (3), we first prove (1) holds with $p = 2$. For $f \in H^2$, it is proved in [3] that

$$| f(z) | \leq 16^2(\tilde{\varphi}(z) + \| \varphi \|_1), \tag{8}$$

here $\phi(t) = f(e^{it})$, and $\tilde{\varphi}(z) = \sup_I \frac{1}{|I|} \int_I |\phi(t)| dt$, where the supremum is taken over all intervals I containing I_z of length $|I| < 1$, and I_z be the boundary arc

$$I_z = \{ e^{it} : \theta - \frac{1}{2}(1 - r) \leq t \leq \theta + \frac{1}{2}(1 - r) \}$$

for each point $z = re^{i\theta} \neq 0$ in U.

Therefore, by (8), elementary inequality $(a + b)^2 \leq 2(a^2 + b^2)$ and the downward convexity of ϕ,

$$\int_U \phi(| f(z) |^2) d\mu(z) \leq \int_U \phi\{[16^2(\tilde{\varphi}(z) + \| \varphi \|_1)]^2\} d\mu$$

$$=\frac{1}{2}\left\{\int_U \phi(c_1\widetilde{\varphi}(z)^2)^2 d\mu+\int_U \phi(c_1\|\varphi\|_2^2)d\mu\right\} \tag{9}$$

with $c_1=512\pi^4$.

It suffices then to show that

$$\int_U \phi(c_1\widetilde{\varphi}(z)^2)d\mu<\phi(c_2\|\varphi\|_2^2) . \tag{10}$$

To do this, let

$$E_s=\{z\in U:\widetilde{\varphi}(z)>s/\sqrt{c_1}>0\},\quad a(s)=\mu(E_s),$$

then

$$\int_U \phi(c_1\widetilde{\varphi}(z)^2)d\mu=-\int_0^\infty \phi(s^2)da(s)$$

$$<2\int_0^\infty s\phi'(s^2)a(s)ds+a(s)\phi(s^2)|_{s=0} . \tag{11}$$

We will show that

$$\lim_{s\to 0}a(s)\phi(s^2)=0. \tag{12}$$

Let $\varphi(t)\in L^1(\partial U)$, and let for $\varepsilon>0$,

$$A_s^\varepsilon=\left\{z\in U:\int_{I_z}|\varphi(t)|dt>s(\varepsilon+I_z)\right\} ,$$

$$B_s^\varepsilon=\left\{z\in U:\text{exists } w\in A_s^\varepsilon \text{ such that } I_w\supset I_z\right\} .$$

It is proved in [4] that

$$E_s=\lim_{\varepsilon\to 0}B_s^\varepsilon ,\qquad \mu(E_s)=\lim_{\varepsilon\to 0}(B_s^\varepsilon) , \tag{13}$$

and there exists a finite number of points $z_1, z_2, \cdots, z_m$ in U such that the arcs I_{z_n} are disjoint and

$$B_s^\varepsilon\subset\bigcup_{n=1}^m\{z\in U:I_z\subset I_{z_n}\} ,$$

and

$$s\sum_{n=1}^m(\varepsilon+|I_{z_n}|)<\sum_{n=1}^m\int_{J_{z_n}}|\varphi(t)|dt<2\pi\|\varphi\|_1, \tag{14}$$

where J_z is the arc of length $5|I_z|$ whose center coincides with that of I_z.

Therefore, since $\mu(S_h)<\phi(Ah)$, by (6) and (14),

$$\mu(B_s^\varepsilon)<\sum_{n=1}^m\mu(\{z\in U:I_z\subset J_{z_n}\})<\sum_{n=1}^m\mu(S_{J_{z_n}})$$

$$<\sum_{n=1}^m\phi(5A|I_{z_n}|)<\phi\Big(\sum_{n=1}^m 5A|I_{z_n}|\Big)<\phi(10\pi A\|\varphi\|_1/s) .$$

Letting $\varepsilon\to 0$, it follows from (13) that

$$a(s)=\mu(E_s)<\phi(A_1\|\varphi\|_1/s) \tag{15}$$

with $A_1 = 10\pi A$.

Thus, $a(s)\phi(s^2) < \phi(A_1\|\varphi\|_1/s)\phi(s^2)$, and (12) holds by condition (iv) of ϕ. Then from (11) we have

$$\int_U \phi(c_1\tilde{\varphi}(z)^2)d\mu(z) < \int_0^s \phi'(s^2)a(s)ds^2 \ . \quad (16)$$

For $s>0$, let

$$\psi_s(t) = \begin{cases} \varphi(t) & \text{wherever } |\varphi(t)|>s/2A_1, \\ 0 & \text{otherwise}. \end{cases}$$

Here we assume $A_1>1$. Let $\tilde{\psi}_s(z) = \sup_I \frac{1}{|I|}\int_I |\psi_s(t)|dt$ defined as $\tilde{\varphi}(z)$, and let $F_s = \{z\in U : \tilde{\psi}_s(z) > s/2\sqrt{c_1} > 0\}$, then it is proved in [4] that $E_s \subset F_s$. Therefore, from (16) and (15) for ψ_s we obtain

$$\int_U \phi(c\tilde{\varphi}(z)^2)d\mu(z) < \int_0^\infty \phi'(s^2)\mu(F_s)ds^2$$

$$< \int_0^\infty \phi'(s^2)\phi(A_1\|\psi_s\|_1/s)ds^2 \ . \quad (17)$$

Since $\phi(t)/t$ is increasing in $(0,\infty)$, then

$$\phi(A_1\|\psi_s\|_1/s) = \left[\phi\left(\frac{A_1}{2\pi s}\int_{2A_1|\varphi(t)|>s}|\varphi(t)|dt\right) \Big/ \left(\frac{A_1}{2\pi s}\int_{2A_1|\varphi(t)|>s}|\varphi(t)|dt\right)\right] \cdot$$

$$\left(\frac{A_1}{2\pi s}\int_{2A_1|\varphi(t)|>s}|\varphi(t)|dt\right)$$

$$< \left\{\phi\left(\frac{A_1}{2\pi s}\int_{2A_1|\varphi(t)|>s}(2A_1|\varphi(t)|^2/s)dt\right) \Big/\right.$$

$$\left.\left(\frac{A_1}{2\pi s}\int_{2A_1|\varphi(t)|>s}(2A_1|\varphi(t)|^2/s)dt\right)\right\} \cdot$$

$$\left\{\frac{A_1}{2\pi s}\int_{2A_1|\varphi(t)|>s}|\varphi(t)|dt\right\}$$

$$< \frac{\phi(2A_1^2\|\varphi\|_2^2/s^2)}{2A_1^2\|\varphi\|_2^2/s^2} \cdot \left\{\frac{A_1}{2\pi s}\int_{2A_1|\varphi(t)|>s}|\varphi(t)|dt\right\}.$$

Substituting this inequality into (17) we have

$$\int_U \phi(c_1\tilde{\varphi}(z)^2)d\mu$$

$$< \frac{1}{2A_1\|\varphi\|_2^2}\int_0^\infty \phi'(s^2)\phi(2A_1^2\|\varphi\|_2^2/s^2)s\left(\frac{1}{2\pi}\int_{2A_1|\varphi(t)|>s}|\varphi(t)|dt\right)ds^2$$

Exchanging the order of integration, since $\sup_{t>0} t\phi'(t)\phi(ct^{-1}) < B\phi(c)$, we have

$\int_U \phi(c_1\tilde{\varphi}(z)^2)d\mu(z) < 2B\phi(2A_1^2\|\varphi\|_2^2)$, and (10) holds with $c_2 = 4A_1^2B$.

Substituting (10) into (9), we have

$$\int_U \phi(|f(z)|^2)\,d\mu(z) < \phi(c_2\|\varphi\|_2^2) + \phi(c_1\|\varphi\|_2^2)\cdot\mu(U)$$
$$< \phi(c\|\varphi\|_2^2) = \phi(c\|f\|_2^2)\ . \tag{18}$$

This proves the sufficiency of (2) for $p=2$.

Finally, for arbitrary p, $0<p<\infty$, if $f\in H^p$, then $f(z)=B(z)[g(z)]^{2/p}$, where $B(z)$ is the Blaschke product and $g(z)\neq 0, g\in H^2$ and $\|f\|_p^p=\|g\|_2^2$. Therefore,

$$\int_U \phi(|f(z)|^p)\,d\mu(z) < \int_U \phi(|g(z)|^2)\,d\mu(z)\,.$$

Since $\mu(S_h)<\phi(Ah)$ for every set S_h of the form (3), then by (18) we have

$$\int_U \phi(|f(z)|^p)\,d\mu(z) < \phi(c\|g\|_2^2) = \phi(c\|f\|_p^p),$$

and this completes the proof of the theorm 1.

Apply theorem 1 to $\phi(t)=t^{q/p}$, $0<p<q<\infty$, we obtain the following corollaries immediately.

Corollary 1.1 Let μ be a finite, positive measure on U, and suppose $0<p<q<\infty$. Then in order that

$$\{\int_U |f(z)|^{qp'/p}\,d\mu(z)\}^{p/q} = c\|f\|_{p'}^{p'} \tag{19}$$

for all $f\in H^{p'}$, $0<p'<\infty$, it is necessary and sufficient that $\mu(S_h)<(Ah)^{q/p}$ for every set S_h of the form (3).

As in [3], [4] two inequalities follow immediately from above corollary 1.1.

Corollary 1.2 If $0<p<q<\infty$, then $f\in H^p$, $0<p<\infty$, implies

$$\{\int_0^1 (1-r)^{q/p-2} M_a^a(r, f)\,dr\}^{p/q} < c\|f\|_{p'}^{p'}, \tag{20}$$

where $a=qp'/p$ and $M_a^a(r,f)=\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^a d\theta$. This generalized a theorem of Hardy-Littlewood [6]:

$$\{\int_0^1 (1-r)^{q/p-2} M_q^q(r, f)\,dr\}^{1/q} < c\|f\|_p\,.$$

Corollary 1.3 If $0<p<q<\infty$, and $f\in H^{p'}$, $0<p'<\infty$, then

$$\{\int_{-1}^1 (1-r)^{q/p-1} |f(r)|^{qp'/p}\,dr\}^{p/q} < c\|f\|_{p'}^{p'}\ . \tag{21}$$

Particularly, for $p'=p$ all of these corollaries reduce to that of Duren in [3].

3. Extension of the theorem of Hastings

Let $U^n=\{z=(z_1,\cdots,z_n)\in C^n: |z_j|<1, 1<j<n\}$ and let σ_n be $2n$-dimensional Lebesgue measure restricted to U^n, normalized so that U^n has measure one.

Theorem 2 Let μ be a finitie, positive measure on U^n, and suppose that function $\phi(t)$ satisfies the first conditions (i)—(iii) of theorem 1, and there exists a constant K such that

$$\phi(t_1)\cdot\phi(t_2)<K\phi(t_1t_2) \tag{22}$$

for arbitrary $t_1, t_2>0$. Then in order that there exists a constant $c>0$ such that

$$\phi^{-1}\{\int_{U^n}\phi(|f(z)|^p)d\mu(z)\}<c\|f\|^p_{A^p}=c\int_{U^n}|f(z)|^p d\sigma_n(z) \tag{23}$$

for all $f\in A^p(U^n)$, $0<p<\infty$, it is necessary and sufficient that there is a constant $A>0$ such that

$$\mu(S_h)<\phi(A\prod_{j=1}^{n}h_j^2) \tag{24}$$

for every set S_h of the form

$$S_h=\{z=(r_1e^{i\theta_1},\ \cdots,\ r_ne^{i\theta_n}):1-h_j<r_j<1,$$
$$\theta_j^0=\theta_j=\theta_j^0+h_j,\ 1\leqslant j\leqslant n\}. \tag{25}$$

Proof If inequality (23) holds for $f\in A^p(U^n)$, then for every set S_h of the form (25) we have

$$\mu(S_h)<\phi(c\|f\|^p_{A^p})/\phi(\min_{z\in S_h}|f(z)|^p). \tag{26}$$

We assume $f(z)=[c_1\prod_{j=1}^{n}h_j^4(1-\bar a_jz_j)^{-4}]^{1/p}$, where $a_j=(1-h_j)\exp\{i(\theta_j^0+h_j)/2\}$, $1<j<n$, then

$$\|f\|^p_{A^p}<c_1\prod_{j=1}^{n}h_j^2,\quad |f(z)|^p>1,\ z\in S_h.$$

Therefore

$$\mu(S_h)<\phi(cc_1\prod_{j=1}^{n}h_j^2)/\phi(1)$$

and (24) holds with $A=cc_1/\phi(1)$ for $\phi(1)<1$ and $A=cc_1$ for $\phi(1)>1$.

Conversely, suppose that (24) holds for every set S_h of the form (25). For $m=(m_1,\cdots,m_n)\in Z^n$ and $k=(k_1,\cdots,k_n)\in Z^n$ with $m_j>0$ and $1<k_j<2^{m_j+4}$, $1<j<n$, set $T_{mk}=\{z=(r_1e^{i\theta_1},\ \cdots,\ r_ne^{i\theta_n}):1-2^{-m_j}<r_j<1-2^{-m_j-1},\ 2k_j\pi/2^{m_j+4}<\theta_j<2(k_j+1)\pi/2^{m_j+4}1<j<n\}$, and let $z^{mk}=(z_1^{mk},\cdots,z_n^{mk})$, where $z_j^{mk}=(1-2^{-m_j})\cdot\exp\{2(k_j+1/2)\pi i/2^{m_j+4}\}$, $1<j<n$, and $U_{mk}=\{z=(z_1,\cdots,z_n)\in C^n: |z_j-z_j^{mk}|<(7/8)2^{-m_j},\ 1<j<n\}$, it is proved in [5] that

$$|f(z)|^p < c_2\Big(\prod_{j=1}^{n} 4^{m_j}\Big)\int_{U_{mk}} |f(z)|^p d\sigma_n(z), \quad z \in T_{mk} \tag{27}$$

and

$$\sum_m \sum_k \int_{U_{mk}} |f(z)|^p d\sigma_n(z) < N \int_{U^n} |f(z)|^p d\sigma_n(z), \tag{28}$$

where $N = (135)^n$. Therefore by (27)

$$\int_{U^n} \phi(|f(z)|^p) d\mu(z) = \sum_{\substack{m=(m_1,\ldots,m_n)\\ m_j > 0}} \ \sum_{\substack{k=(k_1,\ldots,k_n)\\ 1<k_j<2^{m_j+4}}} \int_{T_{mk}} \phi(|f(z)|^p) d\mu(z)$$

$$< \sum_m \sum_k \mu(T_{mk}) \phi\Big\{c_2 \prod_{j=1}^{n} 4^{m_j} \int_{U_{mk}} |f(z)|^p d\sigma_n(z)\Big\}. \tag{29}$$

Since $T_{mk} \subset S_{mk}$ which is the set S_h of the form (25) with $h_j = 2^{-m_j}$, $1 < j < n$, then

$$\mu(T_{mk}) < \mu(S_{mk}) = \phi\Big(A \prod_{j=1}^{n} 2^{-2m_j}\Big). \tag{30}$$

Substituting (30) into (29), by (22), (6) and (28), we have

$$\int_{U^n} \phi(|f|^p) d\mu < \sum_m \sum_k \phi\Big(A \prod_{j=1}^{n} 2^{-2m_j}\Big) \phi\Big\{c_2 \prod_{j=1}^{n} 4^{m_j} \int_{U_{mk}} |f|^p d\sigma_n\Big\}$$

$$< K \sum_m \sum_k \phi\Big(Ac_2 \int_{U_{mk}} |f|^p d\sigma_n\Big) < K\phi\Big(Ac_2 N \int_{U^n} |f|^p d\sigma_n\Big)$$

$$< \phi\Big(c \int_{U^n} |f|^p d\sigma_n\Big) = \phi(c \|f\|^p_{A^p})$$

where $c = Ac_2NK$ for $K > 1$ and $c = Ac_2N$ for $K < 1$.

Hence (23) holds for all $f \in A_p(U^n)$. This completes the proof of the theorem 2.

It follows the following corollaries immediately.

Corollary 2.1 Let μ be a finite, positive measure on U^n, and suppose $0 < p < q < \infty$. Then in order that there exists a constant $c > 0$ such that

$$\Big\{\int_{U^n} |f(z)|^{qp'/p} d\mu(z)\Big\}^{p/q} < c \|f\|^{p'}_{A^{p'}} \tag{31}$$

for all $f \in A^{p'}(U^n)$, $0 < p' < \infty$, it is necessary and sufficient that there is a constant $A > 0$ such that

$$\mu(S_h) < \Big(A \prod_{j=1}^{n} h_j\Big)^{2q/p} \tag{32}$$

for every set S_h of the form (25).

Corollary 2.2 Suppose $0 < p < q < \infty$, if $f \in A^{p'}(U)$, $0 < p' < \infty$, then

$$\Big\{\int_{-1}^{1} |f(r)|^q (1-r)^{2q/p-1} dr\Big\}^{p/q} < c \|f\|^{p'}_{A^{p'}(U^n)} \tag{33}$$

and

$$\{\int(1-r)^{2q/p-1}M_a^q(r,f)dr\}^{p/q}<c'\|f\|^p_{A^{p'}(U^n)}, \qquad (34)$$

where $a=qp'/p$, the constnat c and c' are independent of f.

Remark All these corollaries also hold for every positive n-subharmonic functions f in U^n if $1<p<q<\infty$.

4. Another example

Let $\phi(t)=t^a/(1+t)$, $t>0$, it is easy to see that for $a>2$

$\phi(0)=0$, $\phi'(t)>0$, $\phi''(t)>0$; $\phi(t_1)\cdot\phi(t_2)<\phi(t_1t_2)$, t_1, $t_2>0$;

$$\lim_{t\to 0}\phi(ct^{-1})\phi(t^2)=0 .$$

Finally, we show

$$\sup_{t>0}\frac{t\phi'(t)\phi(ct^{-1})}{\phi(c)}=a . \qquad (35)$$

Since

$$\frac{t\phi'(t)\phi(ct^{-1})}{\phi(c)}=\left(\frac{1+c}{t+c}\right)\frac{t[a+(a-1)t]}{(1+t)^2},\quad \frac{1+c}{t+c}=\begin{cases}1/t, & \text{for } 0<t<1,\\ 1, & \text{for } t>1,\end{cases}$$

therefore

$$\frac{t\phi'(t)\phi(ct^{-1})}{\phi(c)}=g(t)=\begin{cases}\dfrac{a+(a-1)t}{(1+t)^2}, & \text{for } 0<t<1,\\[2mm] \dfrac{at+(a-1)t^2}{(1+t)^2}, & \text{for } t>1.\end{cases}$$

In case of $0<t<1$, since $g'(t)<0$, therefore $g(t)\leqslant a$. In case of $t>1$, since $g'(t)>0$, hence $g(t)\leqslant a-1$. So (35) is true.

Thus $\phi(t)=t^a/(1+t)$ for $a\geqslant 2$ satisfies all the conditions of theorem 1 and 2. Therefore we have the following corollaries.

Corollary 1.4 Let μ be a finite, positve measure on U, anll suppose $a>2$. Then in order that

$$\int_U\frac{|f(z)|^{pa}}{1+|f(z)|^p}d\mu(z)<c\frac{(\|f\|_p^p)}{1+\|f\|_p^p} \qquad (36)$$

for all $f\in H^p$, $0<p<\infty$, it is necessary and sufficient that

$$\mu(S_h)<A\frac{h^a}{1+h} \qquad (37)$$

for every set S_h of the form (3).

Corollary 2.3 Let μ be a finite, positive measure on U^n, and suppose $a>2$. Then there exists a constant $c>0$ such that

$$\int_{U^n}\frac{|f(z)|^{pa}}{1+|f(z)|^p}d\mu(z)<c\frac{(\|f\|^p_{A^p})^a}{1+\|f\|^p_{A^p}} \qquad (38)$$

for all $f \in A^p(U^n)$, $0<p<\infty$, if and only if there exists a constnat $A>0$ such that

$$\mu(S_h) < A \frac{\left(\prod_{j=1}^{n} h_j^2\right)^a}{1+\prod_{j=1}^{n} h_j^2} \tag{39}$$

for every set S_h of the form (25).

References

[1] Carleson, L., Amer. J. Math., 80(1958), 921-930.

[2] ________, Ann. of Math., (2)76(1962), 547-559.

[3] Duren, P.L., Bull. Amer. Math. Soc., 75(1969), 143-146.

[4] ________, Pure and Appl. Math., Vol. 38, Academic Press, New York, 1970.

[5] Hastings, W.W., Proc. Amer. Math. Soc., 52(1975), 237-241.

[6] Hardy, G. H. and Littlewood, J. E., II, Math. Z., 34(1932), 403-439.

(from 40)

$$|\widetilde{D}_n(f,x) - f(x)| \leq K \left(\frac{x(1-x)}{n+1}\right)^{a/2}$$

holds for $x \in [0,1]$ iff $f \in \mathrm{Lip}^* a$.

Theorem 4 Let $f \in C[0,1]$ and $0<a<2$. Then the following two statements are equivalent:

i) $\|\widetilde{D}_n(f) - f\| = O(n^{-a/2})$; $(n \to +\infty)$;

ii) $\varphi(x)^{a/2} |\Delta_h^2(f,x)| \leq K h^a$; $(x \in [h, 1-h],\ h>0)$, where $\varphi(x) = x(1-x)$.

Reference

[1] J.L.Durrmeyer, Thesé de 3e Cycle, Faculté des sciences, de I'University de paris, 1967.