Journal of Mathematical Research and Exposition
Vol.9, No.1, Feb., 1989

Approximation by Modified Durrmeyer-Bernstein Operators*

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J.L. Durrmeyer[1] defined the approximation process

$$D_n(f,x)=\sum_{k=0}^{n}p_{nk}(x)(n+1)\int_0^1 f(t)\,p_{nk}(t)\,dt;\quad p_{nk}(x)=\binom{n}{k}x^k(1-x)^{n-k},$$

which can be used restorig f if its moments $\int_0^1 f(t)t_k dt$ are given.

Now, we shall modify the Durrmeyer operators by introducing the following approximation process. For $f\in C[0,1]$, let

$$\phi_{nk}(f)=\begin{cases} f(0), & k=0,\\ (n-1)\int_0^1 f(t)p_{n-2,k-1}(t)dt, & 1\leqslant k\leqslant n-1,\\ f(1), & k=n,\end{cases}\quad \text{and } \widetilde{D}_n(f,x)=\sum_{k=0}^{n}\phi_{nk}(f)p_{nk}(x),$$

$x\in[0,1]$. which is called modified Durremeyer oprerators.

In this paper, we study the problem of simultaneous approximation by these operators and give out exact estimate.

The following main results are obtained.

Theorem 1 If $f\in C^r[0,1]$, then, for every $x\in[0,1]$, we have

$$\left|\frac{(n+r-1)!\,(n-1)!}{n!\,(n-1)!}\widetilde{D}_n^{(r)}(f,x)-f^{(r)}(x)\right|\leqslant\frac{3+r}{2}\,\omega\left(f^{(r)},\frac{1}{\sqrt{n}}\right),$$

where $0\leqslant r<n$.

Corollary If $f\in C^r[0,1]$, then, for $0\leqslant k\leqslant r$, the relation

$$\lim_{n\to+\infty}\widetilde{D}_n^{(k)}(f,x)=f^{(k)}(x).$$

holds uniformly on $[0,1]$.

Theorem 2 Assume thar $f^{(r)}$ is integrable and bounded on $[0,1]$ and that the $(r+2)$-th deriative $f^{(r+2)}(x)$ exist, at $x\in[0,1]$. Then

$$\lim_{n\to+\infty}n\left(\widetilde{D}_n^{(r)}(f,x)-f^{(r)}(x)\right)=r(1-2x)f^{(r+1)}(x)+x(1-x)f^{(r+2)}(x).$$

In particalar, if $f\in C^{r+2}[0,1]$, then the above relation holds uniformly on $[0,1]$.

Theorem 3 Let $f\in C[0,1]$ and $0<a<2$. Then the relation (to 39)

* *Received Nov.* 15, 1986.

for all $f\in A^p(U^n)$, $0<p<\infty$, if and only if there exists a constnat $A>0$ such that

$$\mu(S_h)<A\frac{(\prod_{j=1}^{n}h_j^2)^a}{1+\prod_{j=1}^{n}h_j^2} \tag{39}$$

for every set S_h of the form (25).

References

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(from 40)

$$|\widetilde{D}_n(f,x)-f(x)|\leqslant K\left(\frac{x(1-x)}{n+1}\right)^{a/2}$$

holds for $x\in[0,1]$ iff $f\in \mathrm{Lip}^*a$.

Theorem 4 Let $f\in C[0,1]$ and $0<a<2$. Then the following two statements are equivalent:

i) $\|\widetilde{D}_n(f)-f\|=O(n^{-a/2})$; $(n\to+\infty)$;

ii) $\varphi(x)^{a/2}|\Delta_h^2(f,x)|\leqslant Kh^a$; $(x\in[h,1-h],\ h>0)$, where $\varphi(x)=x(1-x)$.

Reference

[1] J.L.Durrmeyer, Thesé de 3e Cycle, Faculté des sciences, de I'University de paris, 1967.