Journal of Mathematical Research and Exposition
Vol.9, No.1, Feb., 1989

Families of Semiopen Sets and Their Accompanied Topologies*

Zhu Jingfeng

(Coal Economical Collage of China, Yantai)

Introduction Let (X, U) be a topological space. A set $s\subset X$ is called a semiopen set of U, if there exists $u\in U$ such that $u\subset s\subset u^-$, where u^- expresses the closure of u. The family $\varphi=\varphi_u$ of all semiopen sets of U is called the family of semiopen sets of U, and the topology U an accompanied topology of φ. The class of all the topologies on X generating the same family φ of semiopen sets is called a class of semi-homeomorphic topologies (in symbol $[\varphi]$), and $F(U)$[2] is the finest topology in $[\varphi]$.

S. G. Crossley, S. K. Hitderad[1]—[3] and Yang Zhongqiang[4] discuss the properties of semi-homeomorphic topologies. This paper investigates the structure of families of semiopen sets and the ralation between the inclusion of families of semiopen sets and the inclusion of their accompanied topologies.

§ 1 The structure of families of semiopensets

Theorem 1.1 A falimy φ of sets in X is a family of semiopen sets of some topology on X if and only if the following and satisfied

Ⅰ $\varphi, X\in\varphi$.

Ⅱ φ is closed with union operation,

Ⅲ $\varphi=\{s\subset X:\ \forall v\in F_\varphi,\ s\cap v\neq\phi,\ \exists u\in F_\varphi\setminus\{\phi\},\ u\subset s\cap v\}$.

where $F_\varphi=\{v\in\varphi:\ \forall s\in\varphi,\ v\cap s\in\varphi\}$.

Proof Necessity. Let φ be a family of semiopen sets of some topology on X, by [4], F_φ is the finest accompanied topology of φ. Therefore, Ⅰ, Ⅱ, Ⅲ hold.

Sufficiency. Clearly, F_φ is a topology on X. We complete the proof by verifying that F_φ is an accompanied topology of φ. For given $s\subset X$ being a semiopen set of F_φ, if $s=\phi$, then $s\in\varphi$. Now assume $s\neq\phi$, then for any $v\in F_\varphi$, $s\cap v\neq\phi$, by [1], $s\cap v$ is also a semiopen set of F_φ, Therefore, there exists a nonempty $u\in F_\varphi$ such that $u\subset s\cap v\subset u^-$. It follows by Ⅲ that $s\in\varphi$.

Conversely, for nonempty $s\in\varphi$, since $s\cap X=s\neq\phi$, by Ⅰ, Ⅲ, there exists a nonempty $u\in F_\varphi$ with $u\subset s\cap X=s$. This shows that the interior s° of s is a nonempty

*Received Dec. 31, 1986.

open set of F_φ. It only remains to show $s^\circ \subset s \subset s^{\circ -}$.

For any $x \in s$ and $w \in F_\varphi$ with $x \in w$, since $s \cap w \supset \{x\} \neq \phi$, again by Ⅲ, there exists a nonempty $s \in F_\varphi$ so that $\tilde{s} \subset s \cap w$. This shows that $w \cap s^\circ \neq \phi$. Hence, $s \subset s^{\circ -}$ implying that s is a semiopen set of F_φ.

With the same method in Theorem 1.1, we may obtain

Theorem 1.2 Let (X, U) be a topological space, then $s \subset X$ is a semiopen set of U if and only if $\forall v \in U$, $v \cap s \neq \phi$, $\exists u \in U \setminus \{\phi\}$ satisfying $u \in s \cap v$.

Theorem 1.1 shows that a family of semiopen sets in X could be defined by the operation of sets without any topological structure on X previously given.

We introduce a new concept.

Definition 1.3 We say a topology τ thick-contacts a family φ of semiopen sets (a topology U) or τ is a thick-contacting topology of $\varphi(U)$, if each nonempty element of $\varphi(U)$ contains a nonempty element of τ.

It is not difficult to verify the following gemi-topological property:

Proposition 1.4 Let φ and $\tilde{\varphi}$ be families of semiopen sets in X, If there exists some $U_\varphi \in [\varphi]$ thick-contacting φ or some $U_{\tilde{\varphi}} \in [\tilde{\varphi}]$, then all topologies in $[\varphi]$ thick-contact $\tilde{\varphi}$ and all topologies in $[\tilde{\varphi}]$.

§ 2 **Families of semiopen sets.** From now on, we always denote by φ $(\tilde{\varphi})$ a family of semiopen sets in X, by U (V) a topology on X and by φ_τ the family of semiopen sets in X of topology τ.

Lamma 2.1 If $\exists U_\varphi \in [\varphi]$ satisfying $U_\varphi \subset V \subset \varphi$, then $\varphi_V \subset \varphi$.

Proof Let $s \in \varphi_V$. Since $U_\varphi \subset V$, by Thegrem 1.2, for any $v \in U_\varphi$, $s \cap v \neq \phi$, there exists nonempty u in $V \subset \varphi$ such that $u \subset s \cap v$. Notice nonempty $u \in \varphi$ implying u is a semiopen set of U_φ, $\exists w \in U_\varphi \setminus \{\phi\}$ with $w \subset u \subset s \cap v$. It follows by Theorem 1.2 that $s \in \varphi$.

Theorem 2.2 (i) $F_\varphi \subsetneqq V \subsetneqq \varphi$ implies $\varphi_V \subsetneqq \varphi$.

(ii) If $V \not\subset F_\varphi$, $F_\varphi \not\subset V$ and $\exists U_\varphi \in [\varphi]$ such that $U_\varphi \subset V \subset \varphi$, then $\varphi_V \subsetneqq \varphi$.

(iii)[2] If $\exists U_\varphi \in [\varphi]$ with $U_\varphi \subset V \subset F_\varphi$, then $\varphi_V = \varphi$.

Proof (i) Utilize Lemma 2.1 and observe that V is not semi-homeomorphic to F_φ.

(ii) By Lemma 2.1 and $\varphi_V \neq \varphi$.

(iii) is an immediate consequece of Lemma 2.1 and Theorem 1.2.

Lemma 2.3 $U \subset V$ implies $V \subset \varphi_U$ iff there exists a topology $\tau \subset U$ thick-contacting V.

Proof If $U \subset V \subset \varphi_U$, then by Definition 1.3, U thick-contacts V. Conversely, suppose $U \subset V$ and topology $\tau \subset U$ thick-contacts V. For each $s \in V$, $v \in U \subset V$, $s \cap v \neq \phi$, then $s \cap v$ is a nonempty open set of V. Notice that $\tau \subset U$ thick-contacts V implies that U thick-contacts V, there exists a nonempty $u \in U$ such that $u \subset s$

$\cap v$. Consequntly, by Theorem 1.2, s is a semiopen set of U or $s\in\varphi_U$.

Theorem 2.4 If $F_\varphi\subsetneqq F(V)$ and there exists $U_\varphi\in[\varphi]$ thick-contacting V, then $F_\varphi\subsetneqq\varphi_V\subsetneqq\varphi$.

Proof Clearly $F_\varphi\subsetneqq\varphi_V$. Since U_φ thick-contacts V, U_φ thick-contacts $F(V)$. Observe $U_\varphi\subset F_\varphi$ and $F_\varphi\subset F(V)$, by Le mma 2.3, $F(V)\subset\varphi$. It follows by Lemma 2.1 that $\varphi_V\subset\varphi$ If $\varphi_V=\varphi$, then $F_\varphi=F_{\varphi_V}=F(V)$ contradicting the condition $F_\varphi\neq F(V)$. Thus, $\varphi_V\subsetneqq\varphi$.

Note 1 The conditions in Theorem 2.4 are also necessary (see Theorem 3.1).

Lamma 2.5 Suppose $V\subset\varphi$. If $V\not\subset F_\varphi$, then $\varphi\not\subset\varphi_V$.

Proof Choose $s\in V\setminus F_\varphi$, denoet $\tilde{s}=s^{-/[F_\varphi]}\cup(s\setminus s^{\circ[F_\varphi]})$ where $s^{-/[F_\varphi]}(s^{\circ[F_\varphi]})$ denotes the complement (interior) of $s^-(s)$ with respect to topology F_φ. It is obvious that $\tilde{s}\in\varphi$ since $s\in\varphi$. We complete the proof by showing that $\tilde{s}\notin\varphi_V$. In fact, since $s\notin F_\varphi$, $s\cap\tilde{s}=s\setminus s^{\circ[F_\varphi]}\neq\phi$. containing no nonempty open set in F_φ. Therefore, $s\cap\tilde{s}$ containing no nonempty set in V since $V\subset\varphi$. Thus, by Theorem 1.2, $\tilde{s}\notin\varphi_V$.

Theorem 2.6 Assume $F(V)\subsetneqq\varphi$, then the conditions (a) $F(V)\not\subset F_\varphi$, $F_\varphi\not\subset F(V)$ and (b) V thick-contacts some $U_\varphi\in[\varphi]$ imply $F_\varphi\subset\varphi_V$, $\varphi_V\not\subset\varphi$ and $\varphi\not\subset\varphi_V$.

Proof The assertion $\varphi\not\subset\varphi_V$ follows immediately from Lemma 2.5 since $\varphi_{F(V)}=\varphi_V$. If $F_\varphi\subset\varphi_V$ then the relation of F_φ, φ_V and $F_{\varphi_V}=\varphi$ will satisfy the condition of Lemma 2.5, therefore, we will have $\varphi_V\not\subset\varphi_{F_\varphi}=\varphi$. Thus, to finish the proof, it is sufficient to show $F_\varphi\subset\varphi_V$.

For given $s\in F_\varphi$, $v\in F(V)\subset\varphi$, $s\cap v\neq\phi$, by Theorem 1.2, there exists nonempty $\tilde{u}$ in F_φ such that $u\subset s\cap v$. Since V thick-contacts U_φ, there exists nonempty $\tilde{u}$ in $F(V)$ with $\tilde{u}\subset u\subset s\cap v$. It follows by Theorem 1.2 that $s\in\varphi_V$. This means that $F_\varphi\subset\varphi_V$.

Note 2 The conditions (a) and (b) in Theorem 2.6 are also nenessary (see Theorem 3.2).

Theorem 2.7 Suppose 1° $V\subsetneqq F_\varphi$; 2° V contains no accompanied topology of φ and 3° V thick-contacts some $U_\varphi\in[\varphi]$. Then (1) $\varphi\subsetneqq\varphi_V$ and (2) $F(V)\subsetneqq\varphi$.

Proof By 3° and Proposition 1.4, V thick-contacts F_φ. It follows by 1° and Lemma 2.3 that $F_\varphi\subset\varphi_V$. Consequently, from Lemma 2.1, we have $\varphi=\varphi_{F_\varphi}\subset\varphi_V$. Therefore, by 2°, (1) holds.

For $s\in F(V)$, $v\in F_\varphi$, $s\cap v\neq\phi$, since $F_\varphi\subset\varphi_V$, by Theorem 1.2, there exists a nonempty u in $F(V)$ such that $u\subset s\cap v$. Hence, there exists a nonempty $\tilde{u}\in V\subset F_\varphi$ such that $\tilde{u}\subset u\subset s\cap v$. It follows by Theorem 1.2 that $s\in\varphi_{F_\varphi}=\varphi$. Thus, $F(V)\subset\varphi$. Again by 2°, we obtain $F(V)\subsetneqq\varphi$.

Note 3 In Theorem 3.3, we will show that the conditions 1°, 2° and 3° in Theorem 2.7 are also neoessary.

§ **3 Accampanied topologies** Throughout this section, we always denote by

U_φ and $U_{\tilde{\varphi}}$ two accompanied topologies of φ and $\tilde{\varphi}$ respectively.

Theorem 3.1 Suppose $F_\varphi \subsetneqq \tilde{\varphi} \subsetneqq \varphi$, then (i) $U_{\tilde{\varphi}} \not\subset F_\varphi$, (ii) $F_\varphi \subsetneqq F_{\tilde{\varphi}}$ and (iii) U_φ thick-contacts $U_{\tilde{\varphi}}$.

Proof (i) If $U_{\tilde{\varphi}} \subset F_\varphi$, then by Lemma 2.1, $\varphi = \varphi_{F_\varphi} \subset \tilde{\varphi}$, a contradiction.

(ii) If $F_\varphi \not\subset F_{\tilde{\varphi}}$, then by Lemma 2.5, $\tilde{\varphi} \not\subset \varphi_{F_\varphi} = \varphi$, also a contradiction.

(iii) follows by $F_\varphi \subset \tilde{\varphi}$ and Proposition 1.4.

Theorem 3.2 Let $F_{\tilde{\varphi}} \subseteq \varphi$, then conditions 1° $\varphi \not\subset \tilde{\varphi}$, $\tilde{\varphi} \not\subset \varphi$ and 2° $F_\varphi \subset \tilde{\varphi}$ imply (i) $F_{\tilde{\varphi}} \not\subset F_\varphi$, $F_\varphi \not\subset F_{\tilde{\varphi}}$, (ii) $U_{\tilde{\varphi}}$ (U_φ) thick-contacts U_φ ($U_{\tilde{\varphi}}$).

Proof (i) If $U_{\tilde{\varphi}} \subset F_{\tilde{\varphi}}$, then by Lamma 2.1, $\varphi = \varphi_{F_\varphi} \subset \tilde{\varphi}$ contradicting 1°. The fact $U_\varphi \not\subset F_{\tilde{\varphi}}$ is verified similarly. Hence, (i) holds.

(ii) follows immediately from $U_\varphi \subset F_\varphi \subset \tilde{\varphi}$ and $U_{\tilde{\varphi}} \subset F_{\tilde{\varphi}} \subset \varphi$ and Proposition 1.4.

Theorem 3.3 Assume $\varphi \subsetneqq \tilde{\varphi}$ and that there exists $U_{\tilde{\varphi}} \subseteq \varphi$, then (i) $U_{\tilde{\varphi}} \subseteq F_\varphi$, (ii) $U_{\tilde{\varphi}}$ contains no accompanied topology of φ and (iii) $U_{\tilde{\varphi}}$ thick-contacts U_φ.

Proof (i) follows directly from Lamma 2.5.

(ii) Observe $\varphi \subset \tilde{\varphi}$ and (iii) of Theorem 2.2, clearly $U_\varphi \not\subset U_{\tilde{\varphi}}$ for any $U_\varphi \in [\varphi]$.

(iii) Since $U_{\tilde{\varphi}} \subset F_\varphi \subset \tilde{\varphi}$, by Lemma 2.3, there exists a topology $\tau \subset U_{\tilde{\varphi}}$ thick-contacting U_φ. Therefore, $U_{\tilde{\varphi}}$ thick-contacts U_φ.

References

[1] S. G. Crossley and S. K. Hitderad, Fund. Math., 74(1972), 233—253.

[2] S. G. Crossley, Proc. Amer. Math. Soc., 43(1974), 416—420.

[3] S. G. Crossley, Proc. Amer. Math. Soc., 72(1978), 409—412.

[4] Yang Zhongqiang, Chinese Scientia Sinica, 7(1984), 388—390.