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Minimum Norm Reflexive Generalized Inverse A_{mr}^- and Least-Squares Reflexive Generalized Inverse A_{lr}^-

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There are many applications of minimum norm reflexive generalized inverse A_{mr}^- and least-squares reflexive generalized inverse in practice. This paper is devoted to the development of their theoretical properties, reverse order law, representation and general expression.

In this paper R(A) denotes the range of A, N(A) denotes the null space of A, $r(A)$ denotes the rank of A, $n(A)$ denotes the nullity of A, $P_{R(A)}$ denotes the orthogonal projector on R(A), $P_{S,T}$ denotes the oblique projector on subspace S along subspace T.

An n by m matrix G is called as Moore-Penrose inverse if G satisfies the four Penrose equations

(1) $$AGA = A$$
(2) $$GAG = G$$
(3) $$(AG)^* = AG$$
(4) $$(GA)^* = GA \quad .$$

An n by m matrix G satisfying the ith, jth, and kth equations in (1)-(4) is called an (i, j, k)-inverse of A.

Classification

Name	Symbol	Satisfied equation
{1,2}-inverse or reflexive g-inverse	$A^{(1,2,)}$ or A_r^-	(1),(2)
{1,3}-inverse or least-squares g-inverse	$A^{(1,3)}$ or A_l^-	(1),(3)
{1,4}-inverse or minimum norm g-inverse	$A^{(1,4)}$ or A_m^-	(1),(4)
{1,2,3}-inverse or least squares reflexive g-inverse	$A^{(1,2,3)}$ or A_{lr}^-	(1),(2),(3)
{1,2,4}-inverse or minimum norm reflexive g-inverse	$A^{(1,2,4)}$ or A_{mr}^-	(1),(2),(4)

The following are main results.

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Consider the space decomposition

(5) $$C^n = N(A)^{\perp} \oplus N(A), \quad C^m = R(A) \oplus N(G).$$

Then we obtain from definition A^-_{mr},

(6) $$G \in A\{1,2,4\} \Longleftrightarrow \begin{cases} GA = P_{R(A^*)} = P_{R(G)} \\ AG = P_{R(A),\ N(G)} \end{cases}$$

$$\Longleftrightarrow \begin{cases} GAx = x, \quad x \in R(A^*) = R(G) \\ Gy = 0, \quad y \in N(G). \end{cases}$$

It is easy to show that if there exist a nonsingular $m \times m$ matrix R and nonsingular $n \times n$ matrix P such that

(7) $$RAP = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

then

(8) $$A^-_{mr} = P \begin{pmatrix} I_r & U \\ V & W \end{pmatrix} R,$$

where

(9) $$W = VU, \quad P \begin{pmatrix} I_r & 0 \\ V & 0 \end{pmatrix} P^{-1} \text{ is a Hermite matrix.}$$

Furthermore, if

(10) $$R = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \begin{matrix} r \\ m-r \end{matrix}, \quad P = \overset{\quad r \qquad n-r}{[P_1 \quad P_2]},$$

then

(11) $$A^-_{mr} = (P^*)^{-1} \begin{pmatrix} (P_1^* P_1)^{-1} & U \\ 0 & 0 \end{pmatrix} R,$$

where $U \in C^{r \times (m-r)}$ is arbitrary.

Similarly, if we consider the space decomposition

(12) $$C^n = R(G) \oplus N(A), \quad C^m = R(A) + R(A)^{\perp},$$

then we have

(13) $$G \in A\{1,2,3\} \Longleftrightarrow \begin{cases} AG = P_{R(A)} \\ GA = P_{R(G),\ N(A)}. \end{cases}$$

For an inconsistent linear system $Ax = b$, $b \in C^m$, if $b = b_1 + b_2$, where $b_1 \in R(A)$, $b_2 \in R(A)^{\perp}$, then $G \in A\{1,2,3\}$ implies that

(14) $$Gb = G(b_1 + b_2) = Gb_1$$

and

(15) $$Gb_1 = GAx = x, \quad \forall x \in R(G).$$

If A has a decomposition (7), then

(16) $$A^-_{lr} = P \begin{pmatrix} I_r & U \\ V & W \end{pmatrix} R,$$

where

(17) $W = VU, \quad R^{-1}\begin{pmatrix} I_r & U \\ 0 & 0 \end{pmatrix} R$ is a Hermite matrix.

Furthermore, we have

(18) $$A_{lr}^- = P\begin{pmatrix} (R_1R_1^*)^{-1} & 0 \\ V & 0 \end{pmatrix}(R^*)^{-1},$$

where $V \in C^{(n-r)\times r}$ is arbitrary.

By means of the reverse order law of Moore-Penrose inverse and theorem 2.5 of Bouldin[2], it follows that

Reverse order law:

(19) $$(AB)_{mr}^- = B_{mr}^- A_{mr}^- (\text{or } B_{mr}^- A_{mr}^- \in AB\{1,2,4\})$$
$$\Longleftrightarrow \begin{cases} R(BB^*A^*) \subset R(A^*) \\ \text{both } BB_r^- \text{ and } AA_r^- \text{ are commutative.} \end{cases}$$

(20) $$(AB)_{lr}^- = B_{lr}^- A_{lr}^- (\text{or } B_{lr}^- A_{lr}^- \in AB\{1,2,3\})$$
$$\Longleftrightarrow \begin{cases} R(A^*AB) \subset R(B) \\ \text{both } BB_r^- \text{ and } AA_r^- \text{ are commutative.} \end{cases}$$

We also characterize A_{mr}^- and A_{lr}^- without proof:

(21) $$A\{1,2,4\} = \{YZ \mid ZAY = I_r, R(Y) = R(A^*), Y \in C_r^{n\times r}, Z \in C_r^{r\times m}\}$$

(22) $$A\{1,2,3\} = \{YZ \mid ZAY = I_r, N(Z) = N(A^*), Y \in C^{n\times r}, Z \in C_r^{r\times m}\}$$

Finally, we give representations and weighted representations of A_{mr}^- and A_{lr}^-:

(23) $$A_{mr}^- = A_m^- A A^-,$$

(24) $$A_{mr}^- = A^*(AA^*)^-,$$

(25) $$A_{m(N),r}^- = A_{m(N)}^- A A^-,$$

(26) $$A_{m(N),r}^- = N^{-1}A^*(AN^{-1}A^*)^-,$$

(27) $$A_{lr}^- = A^- A A_l^-,$$

(28) $$A_{lr}^- = (A^*A)^- A^*,$$

(29) $$A_{l(M),r}^- = A^- A A_{l(M)}^-,$$

(30) $$A_{l(M),r}^- = (A^*MA)^- A^*M,$$

where $A_{m(N)}^-$ is a minimum N-norm generalized inverse, $A_{l(M)}^-$ is a M-least-squares generalized inverse.

References

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[3] He Xuchu and Sun Wenyu, Introduction to Generalized Inverses of Matrices, Teaching Material of Nanjing University, 1982, Jiangsu Science & Technology Publishing House, Nanjing, China, to be published in 1989.