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A Kind of Cubic C^1—Interpolations in the n-dimensional Finite Element Method*

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§ 1 Introduction

As is well known that the multivariate spline function plays an important role in both theory and application. The paper [1]—[11] hove studied the multivariate spline functions and obtained a lot of results concerning this topic. Especially in [3], the existance theorem has been shown for the case of n-dimentional spline functions. A. Ženíšek [10] and P. Alfeld [11] have established some of results about the tetrahedron partition. In this paper. we will show a kind of cubic C^1—interpolations for any n-simplicial partition in $\mathbf{R}^n$. Of course, some of the subdivisions will be needed.

§ 2 The structure of cubic C^1—interpolation

Let Ω_n be a polyhedron in $\mathbf{R}^n$, $\triangle_n$ a simplicial subdivision of Ω_n, and i-simplex $S_j^{(i)}$, $i=0,1,\cdots,n$; $j=1,2,\cdots,T_i$. Suppose $\overline{\triangle}_n$ is a refining subdivision of $\triangle_n$ formed by the following steps:

i). Take an interior point $O_j^{(i)}$ in each i-simplex $S_j^{(i)}$, respectively, $i=2,3,\cdots,n$; $j=1,2,\cdots,T_i$.

ii). Let $S_{j,0}^{(0)}, S_{j,1}^{(0)},\cdots,S_{j,i}^{(0)}$ be the vertices of the i-simplex $S_j^{(i)}$, we join $O_j^{(i)}$ to each $S_{j,k}^{(0)}$ respectively, $i=2,\cdots,n$; $j=1,2,\cdots,T_i$; $k=0,1,\cdots,i$.

iii). Let $S_j^{(i)}, S_{j,k}^{(0)}$ $(0\leqslant k\leqslant i)$ be the same as ii). $S_{j,i_0}^{(0)}, S_{j,i_1}^{(0)}; S_{j,i_2}^{(0)},\cdots,S_{j,i_m}^{(0)}$ the vertices of the (m)—simplex $S_{j,A}^{(m)}$; and $O_{j,A}^{(m)}$ be the interior point of $S_{j,A}^{(m)}$. We join $O_j^{(i)}$ to each $O_{j,A}^{(m)}$, respectively, where $i=3,\cdots,n$; $j=1,2,\cdots,T_i$; $m=2,3,\cdots,i-1$: and $A=\{i_0,i_1,\cdots,i_m\}\subset\{i=1,2,\cdots,m\}$.

iv). When two n-simplices $S_i^{(n)}$ and $S_j^{(n)}$ have a $(n-1)$—dimentional common surface $S_k^{(n-1)}$, the interior points $O_i^{(n)}, O_j^{(n)}, O_k^{(n-1)}$ have to be collinear.

It is obvious that the refining subdivision $\overline{\triangle}_n$ exists, in fact, we can take

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point $O_i^{(n)}$ as the center of inscribed sphere of $S_i^{(n)}$, where $i=1,2,\cdots,T_n$. When two n simplices $S_i^{(n)}$ and $S_j^{(n)}$ have a common surface $S_k^{(n-1)}$, we set point $O_k^{(n-1)}$ as the intersection point of the surface $S_k^{(n-1)}$ and the straight line $\overline{O_i^{(n)}O_j^{(n)}}$, the others point $O_j^{(i)}$ may be any interior point of $S_j^{(i)}$, respectively.

For the refining subdivision $\overline{\triangle}_n$, we have the interpolation conditions as follows:

1). Values of position, gradient at $S_i^{(0)}$, $i=1,2,\cdots,T_0$.

2). Let $\bar{S}_i^{(1)}$ be the mid-point of $S_i^{(1)}$, e_{i_0} a unit direction vector of $S_i^{(1)}$, and e_{i_j} the vectors sotisfying the following conditions:

$$(e_{i_j}, e_{i_k}) = \delta_{j,k}.$$

where $\delta_{j,k}$ is kronecker symbol, $j,k=0,1,\cdots,n-1$. We give the directional derivatives $\frac{\partial f}{\partial e_{i_j}}(\bar{S}_i^{(1)})$, $j=1,2,\cdots,n-1$; $i=1,2,\cdots,T_1$.

For any simplicial subdivision $\triangle_n$ of a polyhedron domain in $\mathbf{R}^n$, we define

$$S_k^\mu(\triangle_n, \mathbf{R}^n) := \{S \in C^\mu(\triangle_n);\ \text{the restriction of } S \text{ to each } n \text{ simplex of } \triangle_n \text{ is a polynomial of degree } k\}$$

We have

Theorem 1. The interpolation conditions 1) and 2) determine a unigue multivariate spline function $S \in S_3^1(\overline{\triangle}_n, \mathbf{R}^n)$, and

$$\dim S_3^1(\overline{\triangle}_n, \mathbf{R}^n) = (n+1)T_0 + (n-1)T_1.$$

To prove Theorem 1., we need the following three Lemmas.

Lemmas 1. Denote by $V[X_1, X_2, \cdots, X_{n+1}]$ the n-simplex with vertices X_1, $X_2, \cdots, X_{n+1}$. If $X_0 \in V[X_1, \cdots, X_{n+1}]$, then

$$\sum_{i=1}^{n+1} u_i (X_i - X_0) = 0,$$

where $(u_1, u_2, \cdots, u_{n+1})$ is the barycentric coordinates of X_0.

Denote by $D_{i,j} = D_{(X_j - X_i)}$ the (unnormlized) directional derivative of $(X_j - X_i)$, we have

Lemma 2. Let X_1, X_2, X_3 be the vertices of a triangle $\triangle_{123}$, and $P(x)$ a polynomial of degree 3 satisfying the following conditions:

$$P(X_i) = f(X_i),$$
$$D_{i,j}P(X_i) = D_{i,j}f(X_i),\quad i,j=1,2,3.$$

Then

$$P(x) = d u_1 u_2 u_3 + \sum_{i=1}^{3}\Big(\sum_{j=1}^{3} u_j D_{i,j} f(X_i) + (3-2u_i) f(X_i)\Big) u_i^2$$

and

$$D_{i,k}P\Big(\frac{X_i+X_j}{2}\Big) = \frac{1}{4}d + \frac{1}{4}(D_{i,k}f(X_i) + D_{i,k}f(X_j)) - \frac{1}{2}(D_{i,j}f(X_i) + 3f(X_i)),$$

where $i,j,k=1,2,3$; $i \neq j \neq k \neq i$; d is a real constant and the vector (u_1, u_2, u_3)

is the barycentric coordinates of X.

Similarly, we have

Lemma 3. Let X_1, X_2, X_3, X_4 be the vertices of a tetrahedron $V[X_1, X_2, X_3, X_4]$, and $P(x)$ a polynomial of degree 3 satisfying the following conditions:

$$P(X_i)=f(X_i);\ D_{i,j}P(X_i)=D_{i,j}f(X_i),\quad i,j=1,2,3,4.$$

Then

$$P(x)=\sum_{i=1}^{4}[d_i u_i^{-1}\prod_{j=1}^{4}u_j+\left(\sum_{j=1}^{4}u_j D_{i,j}f(X_i)+(3-2u_i)f(X_i)\right)u_i^2],$$

and

$$D_{i,k}P\left(\frac{X_i+X_j}{2}\right)=\frac{1}{4}d_l+\frac{1}{4}(D_{i,k}f(X_i)+D_{i,k}f(X_j))-\frac{1}{2}(D_{i,j}f(X_i)+3f(X_i)),$$

where (i,j,k,l) takes all of the permutations of the four numbers 1, 2, 3, 4; d_1, d_2, d_3, d_4 are real constants, and the vector (u_1, u_2, u_3, u_4) is the bary centric coordinates of X.

The proof of Theorem 1.

Without loss of generality, we will only prove the theorem in the space $\mathbf{R}^3$.

First, Let subdivision $\triangle_3$ have only one tetrahedron $V[X_1, X_2, X_3, X_4]$, and the refining subdivision $\overline{\triangle}_3$ be shown in Fig. 1, we aim at a function $S\in S_3^1(\overline{\triangle}_3, \mathbf{R}^3)$ satisfying the conditions 1) and 2).

First of all, we consider the tetrahedron $V_2:=V[O, X_1, X_2, X_4]$ (See Fig 2.). It is not difficult to verify that

$$\dim S_3^1(\overline{V}_2, \mathbf{R}^3)=22,$$

where $\overline{V}_2$ is the refining subdivision of V_2 which is generated by $\overline{\triangle}_3$.

By using the condition 1) and 2) and the values of position, gradient at point O, we can obtain a unique function $\overline{S}_2\in S_3^1(\overline{V}_2, \mathbf{R}^3)$.

Similarly, we can get the functions $\overline{S}_i\in S_3^1(\overline{V}_i, \mathbf{R}^3)$. where V_i and $\overline{V}_i$ $(i=1,2,3,4)$ are similar to V_2 and $\overline{V}_2$ as above state.

Define the spline function S satisfying $S|V_i=\overline{S}_i$, for each i, it is obvious that $S\in S_3^0(\overline{\triangle}_3, \mathbf{R}^3)$. Next we will get the values of pasition, gradient o S at the point O, such that $S\in S_3^1(\overline{\triangle}_3, \mathbf{R}^3)$.

In fact, in the process has been indicated as above, we used the directional deriatives $D_{0,i}S(O)$ $(1\leqslant i\leqslant 4)$ other than the gradients, where we have supposed O is origin.

In order to get the values of $S(O)$ and $D_{0,i}S(O)$, we set

$$\sum_{i=1}^{4}u_{0,i}D_{0,i}S(O)=0,\quad \sum_{i=1}^{4}u_{0,i}D_{0,i}S\left(\frac{X_j}{2}\right)\Big|_{\triangle_{OX_jX_j}}=O,\quad j=1,2,3,4. \tag{1}$$

where $\triangle_{OX_iX_i}$ represents the segment $\overline{OX_i}$, $D_{0,i}=D_{X_i}$, and the vector $(u_{0,1}, u_{0,2}, u_{0,3}, u_{0,4})$ is the barycentric coordinates of O.

Without loss of generality, suppose the corresponding values at points $(X_i+$

$\mathbf{X}_j)/2$ $(i,j=1,2,3)$ appeared in the conditions 1) and 2) vanish.

According to (1) and Lemma 2, we have

$$\sum_{i=1}^{4} u_{0,i}\mathbf{D}_{0,i}S(\mathbf{O}) = 0,$$

$$3S(\mathbf{O}) + \mathbf{D}_{0,i}S(\mathbf{O}) = 2u_{0,4}(\mathbf{D}_{i,0}f(\overline{i4}) + \frac{1}{4}\mathbf{D}_{0,i}f(\mathbf{X}_4)),$$

$$3S(\mathbf{O}) + \mathbf{D}_{0,4}S(\mathbf{O}) = 2\sum_{i=1}^{3} u_{0,i}(\mathbf{D}_{i,0}f(\overline{i4}) + \frac{1}{2}\mathbf{D}_{0,i}f(\mathbf{X}_4)) + u_{0,4}(3f(\mathbf{X}_4) - \frac{1}{2}\mathbf{D}_{0,4}f(\mathbf{X}_4)).$$

Hence

$$S(\mathbf{O}) = u_{0,4}^2 f(\mathbf{X}_4) - \frac{2}{3}u_{0,4}^2\mathbf{D}_{0,4}f(\mathbf{X}_4) + \frac{4}{3}u_{0,4}\sum_{i=1}^{3} u_{0,i}\mathbf{D}_{i,0}f(\overline{i4}),$$

$$\mathbf{D}_{0,i}S(\mathbf{O}) = 2u_{0,4}(\mathbf{D}_{i,0}f(i4) + \frac{1}{4}\mathbf{D}_{0,i}f(\mathbf{X}_4)) - 3S(\mathbf{O}),$$

$$\mathbf{D}_{0,4}S(\mathbf{O}) = 2\sum_{i=1}^{3} u_{0,i}(\mathbf{D}_{i,0}f(\overline{i4}) + \frac{1}{2}\mathbf{D}_{0,i}f(\ 4\ ))$$
$$+ u_{0,4}(3f(\mathbf{X}_4) - \frac{1}{2}\mathbf{D}_{0,4}f(\mathbf{X}_4)) - 3S(\mathbf{O}), \tag{2}$$

where $$f(\overline{i4}) = f(\frac{\mathbf{X}_i + \mathbf{X}_4}{2});\quad i = 1,2,3.$$

To prove $S\in\mathbf{S}_3^1(\overline{\triangle}_3;\ \mathbf{R}^3)$, we only need to prove the function S belongs to $\mathbf{C}^1$ on the joint of arbitrary two tetrahedrons in $\mathbf{V}_i$ $(1\leqslant i\leqslant 4)$. Using conditions 1) and 2), for example, we have

$$\mathbf{D}_{0,4}\overline{\mathbf{S}}_3(\frac{\mathbf{X}_i + \mathbf{X}_j}{2}) = \mathbf{D}_{0,4}\overline{\mathbf{S}}_4(\frac{\mathbf{X}_i + \mathbf{X}_j}{2}),\quad i,j = 1,2.$$

Moreover, from (1) and Lemma 1., we have

$$\mathbf{D}_{0,4}\overline{\mathbf{S}}_4(\mathbf{O}) = \mathbf{D}_{0,4}\overline{\mathbf{S}}_3(\mathbf{O})$$

and

$$\mathbf{D}_{0,4}\overline{\mathbf{S}}_3(\frac{\mathbf{X}_i}{2}) = \mathbf{D}_{0,4}\overline{\mathbf{S}}_4(\frac{\mathbf{X}_i}{2}),\quad i = 1,2.$$

Thus

$$\mathbf{D}_{0,4}\overline{\mathbf{S}}_3|_{\triangle\mathrm{OX}_1\mathrm{X}_2} = \mathbf{D}_{0,4}\overline{\mathbf{S}}_4|_{\triangle\mathrm{OX}_1\mathrm{X}_2}$$

$|_{\mathbf{V}_3\cup\mathbf{V}_4}\in\mathbf{S}_3^1(\overline{\mathbf{V}}_3\cup\overline{\mathbf{V}}_4,\ \mathbf{R}^3)$. That being so $S\in\mathbf{S}_3^1(\overline{\triangle}_3,\mathbf{R}^3)$.

According to Lemma 3, it is easy to get

$\mathbf{S}(x)|_{\mathbf{V}_4} = (u_{4,1}\mathbf{D}_{0,1}\mathbf{S}(\mathbf{O}) + u_{4,2}\mathbf{D}_{0,2}\mathbf{S}(\mathbf{O}) + u_{4,3}\mathbf{D}_{0,3}\mathbf{S}(\mathbf{O}) + (3 - 2u_{4,0})\mathbf{S}(\mathbf{O}))u_{4,0}^2,$

where vector $(u_{4,1}, u_{4,2}, u_{4,3}, u_{4,0})$ is the barycentric coordinates of $\mathbf{X}\in\mathbf{V}_4$.

Let the barycentrc coordinates of $\mathbf{O}_2$ in the tetrahedron $\mathbf{V}[\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4]$ be the vector $(u_{\mathrm{O}_2,1}0,\ u_{\mathrm{O}_2,3},\ u_{\mathrm{O}_2,4})$. Similarly we can obtain

$$\mathbf{S}(\mathbf{O}_2) = u_{\mathrm{O}_2,4}^2 f(\mathbf{X}_4) - \frac{1}{3}u_{\mathrm{O}_2,4}\mathbf{D}_{\mathrm{O}_2,4}f(\mathbf{X}_4) + \frac{4}{3}u_{\mathrm{O}_2,4}\sum_{i=1}^{3} u_{\mathrm{O}_2,i}\mathbf{D}_{i,\mathrm{O}_2}f(\overline{i4}),$$

$$\mathbf{D}_{\mathrm{O}_2,i}\mathbf{S}(\mathbf{O}_2) = 2u_{\mathrm{O}_2,4}(\mathbf{D}_{i,\mathrm{O}_2}f(\overline{i4}) + \frac{1}{4}\mathbf{D}_{\mathrm{O}_2,i}f(\mathbf{X}_4)) - 3\mathbf{S}(\mathbf{O}_2), \tag{3}$$

$$\mathrm{D}_{O_2,4}\mathrm{S}(O_2)=2\sum_{i=1}^{3}u_{O_2,i}f(\overline{i4})+\frac{1}{2}\mathrm{D}_{O_2,i}f(X_4))+u_{O_2,4}(3f(X_4)-\frac{1}{2}\mathrm{D}_{O_2,4}f(X_4))-3\mathrm{S}(O_2)$$

where $f(\overline{i4})$ are similar to (2); $u_{O_2,2}=0$; $\mathrm{D}_{O_2,i}=\mathrm{D}_{(X_i-O_2)}$ and $i=1,3$.

Suppose
$$f_1=\mathrm{S}\,|\,\mathrm{V}[O,O_2,X_3,X_4],$$
$$f_3=\mathrm{S}\,|\,\mathrm{V}[O,O_2,X_1,X_4],$$
$$f_4=\mathrm{S}\,|\,\mathrm{V}[O,O_2,X_1,X_3].$$

then (f_i-f_j) divides by Π_k^2, where (i,j,k) takes the all of the permutations of the three number 1,3,4, and Π_k is the plane determined by three points O,O_2 and X_k $(k=1,3,4)$. We have

$$\mathrm{D}_{O_2,0}f_1=\mathrm{D}_{O_2,0}f_3=\mathrm{D}_{O_2,0}f_4 \qquad (4)$$

It means that the restriction of $\mathrm{D}_{O_2,0}\mathrm{S}$ to the tetrahedron V_2 is only a polynomial of degree 2, and

$$\mathrm{D}_{O_2,0}\mathrm{S}(X)\,|_{\triangle X_1X_3X_4}=\sum_{i=1}^{4V}\mathrm{D}_{O_2,0}f(X_i)u_{2,i}(2u_{2,i}-1)$$
$$+4(u_{2,1}u_{2,3}\mathrm{D}_{O_2,0}f(\overline{13})+u_{2,3}u_{2,4}\mathrm{D}_{O_2,0}f(\overline{34})+u_{2,1}u_{2,4}\mathrm{D}_{O_2,0}f(\overline{14}))$$

where the vector $(u_{2,1},u_{2,2},u_{2,3},u_{2,4})$ is the barycentric coordinates of $X\in V_2$, and $u_{2,2}=0$. Denote by $F(u_{2,1},u_{2,3},u_{2,4})$ the function defined by the above equality we have

$$\mathrm{D}_{O_2,0}\mathrm{S}(O_2)=F(u_{O_2,1},u_{O_2,3},u_{O_2,4}),$$
$$\mathrm{D}_{O_2,0}\mathrm{S}(\frac{X_1+O_2}{2})=F(\frac{1}{2}+\frac{1}{2}u_{O_2,1},\frac{1}{2}u_{O_2,3},\frac{1}{2}u_{O_2,4}),$$
$$\mathrm{D}_{O_2,0}\mathrm{S}(\frac{X_3+O_2}{2})=F(\frac{1}{2}u_{O_2,1},\frac{1}{2}+\frac{1}{2}u_{O_2,3},\frac{1}{2}u_{O_2,4}), \qquad (5)$$
$$\mathrm{D}_{O_2,0}\mathrm{S}(\frac{X_4+O_2}{2})=F(\frac{1}{2}u_{O_2,1},\frac{1}{2}u_{O_2,3},\frac{1}{2}+\frac{1}{2}u_{O_2,4}).$$

By using Lemma 3, the systems of equations (2), (3), (5), the condition 1) and 2), we can get the explicit expression of S. For example, the expression of S on the tetrahedron $\mathrm{V}[O,O_2,X_1,X_3]$ is

$$\mathrm{S}(x)\,|_{\mathrm{V}[O,O_2,X_1,X_3]}=\sum_{i=0}^{3}[\overline{d}_i\overline{u}_i+\prod_{j=0}^{3}\overline{u}_j+(\sum_{j=0}^{3}\overline{u}_j\mathrm{D}_{i,j}f(A_i)+(3-2\overline{u}_i)f(A_i))\overline{u}_i^2]$$

where $A_0=0$, $A_1=x_1$, $A_2=O_2$, $A_3=x_3$; $\mathrm{D}_{i,j}=\mathrm{D}_{(A_j-A_i)}$; the vector $(\overline{u}_0,\overline{u}_1,\overline{u}_2,\overline{u}_3)$ is the barycentric coordinates of $x\in\mathrm{V}[A_0,A_1,A_2,A_3]$; $\overline{d}_i=4\mathrm{D}_{2,0}S(\frac{A_i+A_2}{2})-$ $(\mathrm{D}_{2,0}S(A_i)+\mathrm{D}_{2,0}f(A_i))+2(\mathrm{D}_{2,i}S(A_2)+3S(A_2))$, $i=1,3$; and $\overline{d}_i=4\mathrm{D}_{3,i}f(\frac{A_1+A_3}{2})$, $i=0,2$.

For general subdivision $\triangle_3$, using the preceding results, we have established an interpolating function S such that $S\,|\,\mathrm{S}_i^{(3)}\in\mathrm{S}_3^1(\overline{\mathrm{S}}_i^{(3)},\mathbf{R}^3)$, $i=1,2,\cdots,T_3$; and $S\in\mathrm{S}_3^0(\overline{\triangle}_3,\mathbf{R}^3)$. In the next place, we will prove when tetrahedrons $\mathrm{S}_i^{(3)}$ and $\mathrm{S}_j^{(3)}$ have a common surface $S_{\bar{k}}^{(2)}$, the function $S \epsilon C^1(S_i^{(3)} \cup S_j^{(3)})$. Let $O_i^{(3)}$, $O_j^{(3)}$, $O_{\bar{k}}^{(2)}$ be collinear, using (4), the restriction both of $D_{\bar{k},i}S_i$ and $D_{\bar{k},j}S_j$ to $S_k^{(2)}$ are polynomials of degree 2. It follows from conditions 1) and 2) that

$$\frac{1}{L_{\bar{k},i}} D_{\bar{k},i} S_i \big|_{S_k} = -\frac{1}{L_{\bar{k},j}} D_{\bar{k},j} S'_j \big|_{S_k^{(2)}} \quad ,$$

where $S_i = S\big|_{S_i^{(3)}}$; $S_j = S\big|_{S_k^{(2)}}$; $D_{\bar{k},i} = D_{(O_i^{(3)} - O_k^{(2)})}$, and $L_{\bar{k},i} = \|O_i^{(3)} - O_k^{(2)}\|$.

Therefore $S \epsilon C^1(S_i^{(3)} \cup S_j^{(3)})$, furthmore $S \epsilon S_3^1(\overline{\triangle}_3, \mathbf{R}^3)$.

It is obvious that the interpolation scheme will reproduce the functions belonging to $S_3^1(\overline{\triangle}_3, \mathbf{R}^3)$, especially, it will reproduce the polynomials $\epsilon \mathscr{P}_3$.

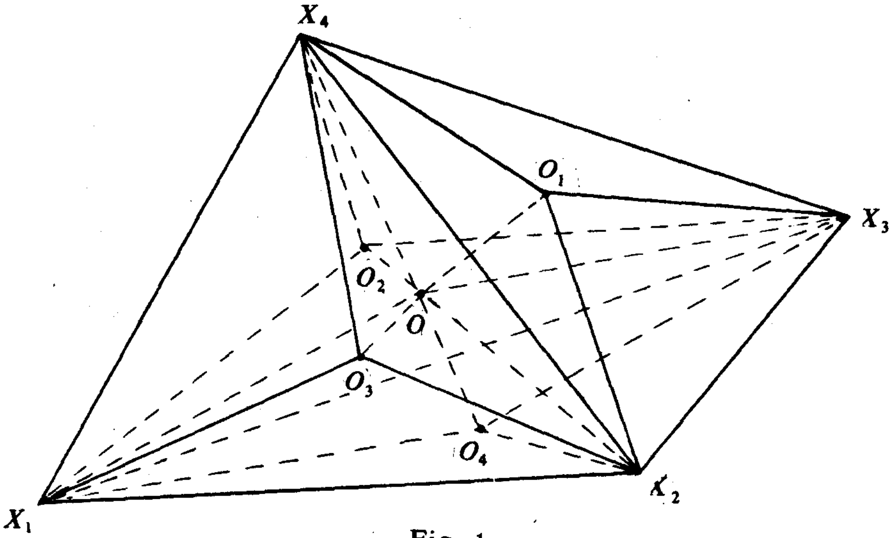

Fig. 1

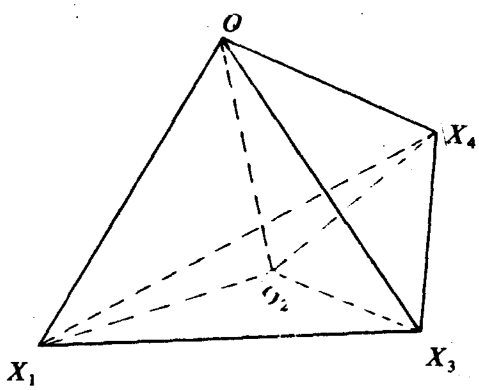

Fig. 2

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