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A Remark on a Paper of Banerjee, Lardy and Lutoborski*

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The paper that may be worthy of remark is entitled "Asymptotic Expansions of Integrals of Certain Rapidly Oscillating Functions" and appeared in Mathematics of Computation, Vol. 49, No. 179(1987), 243—249. What we would like to mention is that the main result of the paper, namely an asymptotic expansion formula with a remainder estimate (cf. § 3), is actually a special consequence of a more general result of the author's paper "Approximate integration of rapidly oscillating functions and of periodic functions" (cf. Proc. Cambridge Phil. Soc. Vol. 59, (1963), 81—88). Here Hsu's result may be cited (§ 2, loc. cit):

Let $F(x, y)$ have continuous partial derivatives with respect to x of orders up to $r \geqslant 1$, and let $F(x, y)$ be periodic of period 1 in y. Then for positive integer λ we have the expansion formula

$$\int_0^1 F(x, \lambda x)\mathrm{d}x = \int_0^1\int_0^1 F\mathrm{d}x\mathrm{d}y + \sum_{j=1}^{r} \frac{\lambda^{-j}}{j!}\int_0^1\left[\frac{\partial^{j-1}F}{\partial x^{j-1}}\right]_{x=0}^{x=1} B_j(y)\mathrm{d}y + \rho_r \qquad (1)$$

where $F \equiv F(x, y)$, $B_j(y)$ is the j-th Bernoulli polynomial and $\rho_r = \rho_r(\lambda)$ is given by

$$\rho_r = -\frac{\lambda^{-r}}{r!}\int_0^1 \mathrm{d}x \int_{\lambda x}^{\lambda x+1} B_r(y - \lambda x)\left(\frac{\partial^r F}{\partial x^r}\right)\mathrm{d}y \qquad (2)$$

Clearly, Banerjee-Lardy-Lutoborski's results are implied by (1) and (2) with $F(x, y) = f(x)\overline{w}(y)$. Indeed, the last term of the summation of (1) and the remainder $\rho_r(\lambda)$ have the same power in λ^{-1}. Thus it is easily seen that the k-term expansion of (1) (with $r = k-1$) leads to the remainder estimate (cf. Corollary 2, loc. cit.).

$$|R_{k,\lambda}| := |\rho_{k-1}(\lambda)| \leqslant 4(2\pi\lambda)^{-k}\cdot\zeta(k)\,\|\partial^k F/\partial x^k\|_1 ,$$

which appears slightly better than that given in B-L-L's paper.

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