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# The Structure of Almost Quasi-Regular Rings*

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Let R be a ring. It is well-known that R is a semigroup with respect to the composition $a\circ b=a+b-ab$ for $a,b\in$ R. An element $a$ of R is said to be (right; left) quasi-regular if there exists an element $b$ in R such that ($a\circ b=0$; $b\circ a=0$) $a\circ b=b\circ a=0$, Denote by Q the set of elements of R which are not quasi-regular in R. We shall use $|S|$ to denote the cardinality of a set S, and we refer to [1] for some results about the cardinal number used in this note.

An infinite ring R is called almost quasi-regular if $|R|>|Q|\geqslant 1$.

In this note, we will prove the following theorem.

**Theorem.** An infinite ring R is almost quasi-regular if and only if $R=eRe+J$, and $eRe$ is a division ring and $|R|>|J|$, where $e$ is a nonzero idempotent and J is the Jacobson radical of R. Moreover, $Q=e+J$.

To prove the theorem, we begin with some lemmas. For convenience, we define $a-B=\{a-b|b\in B\}$ for an element $a$ of R and a subset B of R.

**Lemma 1.** If R is an almost quasi-regular ring with identity element, then R is a division ring.

**Proof.** Let $V$ be the set of elements which are not units of R.

Then $V=1-Q$. Thus $|R|>|Q|=|1-Q|=|V|$. If $V\neq 0$, taken $u\in V$, $u\neq 0$, we can assume that $uR\subset V$. Let $\sim$ be the relation on R difined by $a\sim b$ if $ua=ub$ for $a,b\in$ R. It is easy to verify that $\sim$ is an equivalence relation on R. Denote by $[a]$ the eqivalence class of $a$ under $\sim$ and let $R/\sim$ be the set of equivalence calsses $[a]$. Then $[a]\to ua$ is a 1-1 mapping from $R/\sim$ into $V$, which implies that $|R/\sim|\leqslant|V|$. Since R is infinite, and $|R|>|V|$, we have $|R|=\max\{|[a]|\,|a\in R\}$, whence $|[a]|=|R|>|V|$ for some equivalence class $[a]$. Note that $u(a-[a])=0$, and $u\neq 0$. Then $a-[a]$ contains no unit of R, which implies that $a-[a]\subset V$. It follows that $|[a]|=|a-[a]|\leqslant|V|$. This contridicts the choice of $[a]$. Therefore, $V=0$. Thus R is a division ring.

**Lemma 2.** Let R be an almost quasi-regular ring. Then

(1) R contains nonzero idempotents;

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(2) for every idempotent $e \neq 0$, $eRe$ is a division ring;

(3) every ideal of R is either quasi-regular or almost quasi regular.

**Proof.** Let $u \in Q$. If $u$ is right or left quasi-regular, say $u$ is right quasi-regular, then there exists $v$ in R such that $u \circ v = 0$, whence $v \circ u \circ v \circ u = v \circ u$ and $v \circ u \neq 0$ since $u$ is not quasi-regular. Thus $v \circ u$ is a nonzero idempotent, which is neither right nor left quasi-regular. It follows that Q contains elements which are neither right nor left quasi-regular. For simplicity, we assume that $u$ is neither right nor left quasi-regular. Thus $u \circ R \cup R \circ u \subset Q$. Note that the relation $\sim$ on R difined by $a \sim b$ if $u \circ a = u \circ b$ and $a \circ u = b \circ u$ for $a, b \in R$ is an equivalence relation on R. Denote by $[a, u]$ the equivalence class of $a$ under $\sim$, and let $R/\sim$ be the set of equivalence classes $[a, u]$. Then $[a, u] \to (u \circ a, a \circ u)$ is a 1-1 mapping from $R/\sim$ into $u \circ R \times R \circ u$, from which $|R/\sim| \leqslant |u \circ R \times R \circ u| \leqslant |Q \times Q| = |Q|^2$. Since $|R|$ is infinite and $|R| > |Q|$, we have $|R| > |Q| \geqslant |R/\sim|$, whence $|R| = |[a, u]|$ for some equivalence class $[a, u]$.

Note that $b \in [a, u]$ if and only if $a - b \in [0, u]$, Thus $a - [a, u] \subset [0, u]$, whence $|[0, u]| \geqslant |a - [a, u]| = |[a, u]| = |R|$. Then we have $|[0, u]| = |R|$. Since $|u - [0, u]| = |[0, u]| = |R| > |Q|$, there exists $x \in [0, u]$ such than $u - x$ is quasi-regular. Thus $x \neq 0$ and $u \circ x = x \circ u = u$, that is, $x = ux = xu$. Suppose that $y \in R$ such that $(u - x) \circ y = y \circ (u - x) = 0$. Then we have $u - x + y - uy + xy = 0$. and $x(u - x + y - uy + xy) = 0$, from which $x - x^2 + x^2 y = 0$; that is $x = x^2(u - y)$.Symmetrically, $x = (u - y)x^2$. Observe that $x = x^2(u - y) = x^4(u - y)^3 = x^3(u - y)^3 x^2$. We see that $x(u - y)^3 x^2$ is a nonzero idempotent, since $x \neq 0$. This completes the proof of (1).

Now we prove (2), Let $e$ be a nonzero idempotent of R. Then it is esty to prove that $[0, e] = eRe$ is an infinite ring with identity element $e$. Let $Q(eRe)$ denote the set of elements in R which are not quasi-regular in the ring $eRe$. If $exe$ is quasi-regular in R, then there exists $y$ in R such that $exe \circ y = y \circ exe = 0$, whence $y = exey - exe = yexe - exe \in eRe$. Hence $Q(eRe) \subset Q$. It follows that $|eRe| = |[0, e]| = |R| > |Q| \geqslant |Q(eRe)|$. Consequently, $eRe$ is an almost quasi-regular ring with identity element. By Lemma 1, $eRe$ is a divisioo ring.

Let I be an ideal of R such that $I \not\subset J$, and $Q(I)$ denote the set of elements of I which are not quasi-regular in the ring I. It is clear that $Q(I)$ is not empty. Given $u \in Q(I)$, if $u$ is right or left quasi-regular in R, say right quasi-regular, then there exists $v \in R$ such that $u \circ v = 0$ whence $v = uv - u \in I$, and then $v \circ u$ is a nonzero idempotent contained in I. Hence we can assume that $u$ is neither right nor left quasi-regular in R. For every $x \in [0, u]$, we have $x = ux = xu \in I$. Thus $[0, u] \subset I$, and so $|I| \geqslant |[0, u]| = |R|$. Therefore, $|I| = |R|$ is infinite. Since an element of I is quasi-regular in I if and only if so is it in R, we have $Q(I) \subset Q$. Thus $|I| = |R| > |Q| \geqslant |Q(I)|$, and so I is an almost quasi-regular ring.

**Lemma 3.** Let R be a prime ring with nonzero idempotents such that for every nonzero idempotent $e$, $e\mathrm{R}e$ is a division ring. Then R is a division ring.

**Proof.** It is clear from [2, Lemma 3].

**Lemma 4.** If R is a semiprimitive almost quasi-regular ring, then R is a division ring.

**Proof.** By Lamma 2, R has a nonzero idempotent $e$ and $e\mathrm{R}e$ is a division ring, whence $e\mathrm{R}$ is a minimal right ideal of R, and then $\mathrm{R}e\mathrm{R}$ is a simple ring with minimal right ideals. From Lemma 2 and Lemma 3, we see that $\mathrm{R}e\mathrm{R}$ is a division ring. Thus there exists an ideal T in R such that $\mathrm{R}=\mathrm{R}e\mathrm{R}\oplus\mathrm{T}$. If $\mathrm{T}\neq 0$, T contains a nonzero idempotent $f$ by Lemma 2. Note that $e+f$ is also a non zero idempotent. Then $(e+f)\mathrm{R}(e+f)$ is a division ring by Lemma 2, which contridicts the fact that $ef=0$ and $e, f\in(e+f)\mathrm{R}(e+f)$. Hence $\mathrm{T}\neq 0$, and so $\mathrm{R}=\mathrm{R}e\mathrm{R}$ is a division ring.

**Lemma 5.** Let R be an arbitrary ring with $e^2=e\neq 0$ and the Jacobson radical J such that $e\mathrm{J}e=0$. Then $ere+j$, $j\in\mathrm{J}$, is quasi-regular if and only if $ere$ is quasi-regular (in $e\mathrm{R}e$).

**Proof.** Suppose that $ere+j$ is quasi-regular. Then there exists $y$ in R such that $(ere+j)\circ y=y\circ(ere+j)=0$, from which $e((ere+j)\circ y)e=0$; that is, $ere\circ eye=0$. Similarly, $eye\circ ere=0$. Hence $ere$ is quasi-regular in $e\mathrm{R}e$.

**Conversely.** assume that $ere$ is quasi-regular in $e\mathrm{R}e$. Then exists $ese$ such that $ere\circ ese=ese\circ ere=0$. Since $j\in\mathrm{J}$, there exists $j'$ in J such that $j\circ j'=j'\circ j=0$. Set $k=j'+j'esej'-esej'-j'ese$. Then one can verify that $(ere+j)\circ(ese+k)=(ese+k)\circ(ere+j)=0$. Thus $ere+j$ is quasi-regular.

Now we give the proof of Theorem.

Let R be an almost quasi-regular ring. By Lemma 4, $e+\mathrm{J}\subset\mathrm{Q}$, and then $|\mathrm{R}|>|\mathrm{Q}|\geqslant|e+\mathrm{J}|=|\mathrm{J}|$. Write $\overline{\mathrm{R}}=\mathrm{R}/\mathrm{J}$. Then $|\mathrm{R}|=\max\{|\mathrm{J}|, |\overline{\mathrm{R}}|\}=|\overline{\mathrm{R}}|$, since $|\mathrm{R}|$ is infinite. Denote by $\mathrm{Q}(\overline{\mathrm{R}})$ the set of elements which are not quasi-regular in $\overline{\mathrm{R}}$. Note that $\bar{a}\in\overline{\mathrm{R}}$ is quasi-regular in $\overline{\mathrm{R}}$ if and only if so is $a\in\mathrm{R}$ in R. Then $\mathrm{Q}(\overline{\mathrm{R}})\subset\overline{\mathrm{Q}}$. It follows that $|\overline{\mathrm{R}}|=|\mathrm{R}|>|\mathrm{Q}|=\max\{|\mathrm{J}|, |\overline{\mathrm{Q}}|\}\geqslant|\overline{\mathrm{Q}}|\geqslant|\mathrm{Q}(\overline{\mathrm{R}})|$. Hence $\overline{\mathrm{R}}$ is a semiprimitive almost quasi-regular ring. By Lemma 5, $\overline{\mathrm{R}}$ is a division ring, According to Lamma 2, R has a nonzero idempotent $e$. Then $\mathrm{R}=e\mathrm{R}e+\mathrm{J}$, in which $e\mathrm{R}e$ is a division ring from Lemma 2.

**Conversely.** assume that $\mathrm{R}=e\mathrm{R}e+\mathrm{J}$ and $e\mathrm{R}e$ is a division ring, where $e^2=e\neq 0$, and $|\mathrm{R}|>|\mathrm{J}|$, By Lemma 4, $\mathrm{Q}=e+\mathrm{J}$ and therefore $|\mathrm{R}|>|\mathrm{J}|=|e+\mathrm{J}|=|\mathrm{Q}|$. Since R is an infinite ring, we see that R is an almost quasi-regular ring.

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$$=\frac{\prod_{1\leq i<j\leq n+1}\frac{x_i-x_j}{x_i+x_j}}{x_1+x_2+\cdots+x_{n+1}}\sum_{i=1}^{n+1}x_i\prod_{\substack{j=1\\j\neq i}}^{n+1}\left(\frac{x_i+x_j}{x_i-x_j}\right)$$

对上式的后一和号，结合（2）式的特例：$t=-1$并计算$y$的系数，便有：

$$\sum_{i=1}^{n}x_i\prod_{\substack{j=1\\j\neq i}}^{n}\frac{x_i+x_j}{x_i-x_j}=\sum_{i=1}^{n}x_i$$

由此便知（3）式对$n+1$正确．根据归纳法原理便知对任何自然数$n$,(3)式成立．定理3证毕．

若以$A_n$表示$S_n$的交代子群，$O_n=S_n\setminus A_n$．结合（1）及（3）则有下述推论．

**定理 4**

i．$$\sum_{\sigma\in A_n}\prod_{j=1}^{n}x_{\sigma(j)}\Big/\sum_{i=j}^{n}x_{\sigma(i)}=\frac{1}{2}\left\{1+\sum_{1\leq i<j\leq n}\frac{x_i-x_j}{x_i+x_j}\right\}$$

ii．$$\sum_{\sigma\in O_n}\prod_{j=1}^{n}x_{\sigma(j)}\Big/\sum_{i=j}^{n}x_{\sigma(i)}=\frac{1}{2}\left\{1-\prod_{1\leq i<j\leq n}\frac{x_i-x_j}{x_i+x_j}\right\}$$

若以$S_{\overline{m}}$表示多重集合$\{1^{m_1},2^{m_2},\cdots,n^{m_n}\}$的排列集合，则有命题1的多重形式．

**定理 5** 设$|\overline{m}|=\sum_{i=1}^{n}m_i$，则

$$\sum_{\sigma\in S_{\overline{m}}}\prod_{k=1}^{|\overline{m}|}\frac{x_{\sigma(k)}}{x_{\sigma(k)}+x_{\sigma(k+1)}+\cdots+x_{\sigma(|\overline{m}|)}}=\frac{1}{\prod_{i=1}^{n}m_i!}$$

上式中取$\overline{m}=(1,1,\cdots,1)$便给出命题1.

类似于命题1，定理5的证明可利用归纳法完成，此处从略．

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