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The Unitary Equivalence Problem of Unilateral Weighted Shifts*

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Abstract. In this paper, we study the unilateral weighted shift which is unitarily equivalent to a Toeplitz operator and prove a similar result to that in [1] without the hypothesis that the shift must be hyponormal. As a corollary, we show that if the weight sequence $\{a_n\}_{n=0}^{\infty}$ of the shift is convergent, then

$$1-a_n^2 = (1-a_0^2)^{n+1} \qquad \forall n \geqslant 0$$

1. Introduction. Proof. Sun Shunhua proved in [1] the following result: a hyponormal unilateral weighted shift is unitarily equivalent to a Toeplitz operator if and only if the weight sequence $\{a_n\}_{n=0}^{\infty}$ satisfies

$$1-|a_n|^2 = (1-|a_0|^2)^{n+1} \qquad \forall n \geqslant 0$$

by which we can immediately give the negative answer of question 5 that Halmos rises in 1970 [2], i.e. There is subnormal Teoplitz operator which is neither normal nor analytic. What we are interested in is: Is the hypothesis in [1] that the weighted shift be hyponormal necessary? We shall not consider the trivial case when T is an isometry and consequently ϕ is analytic.

Our main result includes

Theorem 1. If an unilateral weighted shift $T \cong T_\varphi$ $|\varphi|=1$ *a.e.* The weight sequence of T is $\{a_n\}_{n=0}^{\infty}$ and the set $N=\{k: a_k=a_0\}$ is finite, then

$$1-a_n^2 = (1-a_0^2)^{n+1} \qquad \forall n \geqslant 0$$

Corollary 1. If the weight sequence $\{a_n\}_{n=0}^{\infty}$ of the shift T is convergent and $T \cong T_\varphi$ then $1-a_n^2 = (1-a_0^2)^{n+1}$ $\forall n \geqslant 0$

2. Lemmas. Let D be the unit disk in the complex plane, ∂D be the unit circle. $L^\infty(\partial D)$ and $L^2(\partial D)$ be the bounded measurable function space and square integrable function space on the unit circle, $H^\infty(\partial D)$, $H^2(\partial D)$ be ordinary Hardy spaces. We shall denote them by L^∞, L^2, H^∞, H^2.

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Let T be an unilateral weighed shift on H^2 with weight sequence $\{a_n\}_{n=0}^{\infty}$. We can assume $a_n \geqslant 0 \quad \forall n \geqslant 0$ [3]. If T is unitarily equivalent to a Toeplitz operator on H^2, then there exists an orthornormal basis $\{e_n\}_{n=0}^{\infty}$ of H^2 and $\varphi \in L^{\infty}$ such that

$$Te_n = T_\varphi e_n = a_n e_{n+1} \qquad \forall n \geqslant 0$$

Note: Now we have $a_n \neq 0 \ \forall n \geqslant 0$ [4].

Lemma 1. For a fixed k and an element $x \in H^2$, if

$$T_\varphi^* T_\varphi x = a_k^2 x$$

then

$$x = \sum_{i=0}^{\infty} x_i e_{k_i}$$

where k_i satisfy $a_{k_i} = a_k \quad i = 0, 1, 2, \cdots$

Proof. See [5].

Lemma 2. If $|\varphi| = 1$ a.e. then e_0 is an outer function and $\varphi = r(te_0 \sqrt{e_0})$, where r is a constant and $|r| = 1$.

Proof. See [1].

Lemma 3. If $\psi \in L^{\infty}$ and $\psi \in H^2$, then $\psi \in H^{\infty}$.

Proof. See [1].

3. Proof of the theorem.

Theorem 1. If an unilateral weighted shift $T \cong T_\varphi \ |\varphi| = 1$ a.e. the weight sequence of T is $\{a_n\}_{n=0}^{\infty}$ and the set $N = \{k : a_k = a_0\}$ is finite, then

$$1 - a_n^2 = (1 - a_0^2)^{n+1} \qquad \forall n \geqslant 0$$

Proof. Since

$$\varphi e_n = a_n e_{n+1} + (1 - a_n^2)^{1/2} \eta_n \tag{1}$$

$$\overline{\varphi} e_{n+1} = a_n e_n + (1 - a_n^2)^{1/2} \xi_n$$

$$\eta_n, \xi_n \in \overline{H}_0^2 \quad \text{and} \quad \|\eta_n\| = \|\xi_n\| = 1$$

we have

$$te_n = a_n \overline{\varphi} te_{n+1} + (1 - a_n^2)^{1/2} \overline{\varphi} + t_n$$

it is easy to show by computation that

$$T_\varphi^* T_\varphi \overline{t\eta_n} = a_n^2 \overline{t\eta_n}$$

Therefore, by Lamma 1

$$\overline{t\eta_n} = \sum_{i=0}^{\infty} a_i e_{n_i} \quad \text{where} \quad a_{n_i} = a_n \quad i = 0, 1, 2, \cdots$$

Note: We put $e_{n_0} = e_n$.

For the first step, we assume $a_0 \neq a_n \ \forall n > 0$

therefore

$$\overline{t\eta_0} = a_0 e_0 \quad \eta_0 = \overline{a_0} \overline{te_0} \quad |a_0| = 1$$

$$\varphi e_0 = a_0 e_1 + (1 - a_0^2)^{1/2} \overline{a_0} + \overline{e_0}$$

by Lemma 2.

$$\varphi = r \frac{te_0}{\overline{e_0}} \qquad \overline{\varphi} e_0 = \overline{r}\, \overline{te_0}$$

combine these two equalities and by Lemma 3, we have

$$\varphi - \frac{\overline{a_0}}{\overline{r}} (1 - a_0^2)^{1/2} \overline{\varphi} \in H^{\infty}$$

i.e. $$\varphi e_n-\frac{\overline{a_0}}{\bar{r}}(1-a_0^2)^{1/2}\overline{\varphi e_n}$$

$$=a_n e_{n+1}+(1-a_n^2)^{1/2}\eta_n-\frac{\overline{a_0}}{\bar{r}}(1-a_0^2)^{1/2}[a_{n-1}e_{n-1}+(1-a_{n-1}^2)^{1/2}\xi_{n-1}]\in H^2$$

so $$(1-a_n^2)^{1/2}\eta_n-\frac{\overline{a_0}}{\bar{r}}(1-a_0^2)^{1/2}(1-a_{n-1}^2)^{1/2}\xi_{n-1}=0$$

$$1-a_n^2=(1-a_0^2)^{n+1}\quad \forall n\geq 0$$

Before our second step, we need the following two Lemmas.

Lemma 4. If $T_\varphi e_n=Te_n=a_n e_{n+1}$ $\forall n\geq 0$ and $|\varphi|=1$ a.e. then the vector $(\langle\varphi te_0, 1\rangle, \langle\varphi te_{k_1}, 1\rangle\cdots, \langle\varphi te_{k_m}, 1\rangle, \cdots)$ (obviously, it belongs to l^2) can not be a zero vector, where $T_\varphi e_{k_i}=a_{k_i}e_{k_i+1}=a_0 e_{k_i+1}$ and $\langle\cdot\rangle$ represents the inner product in L^2.

Proof. Put $n=0$ in (1) we have

$$\varphi\begin{bmatrix}e_0\\ e_{k_1}\\ \vdots\\ e_{k_m}\\ \vdots\end{bmatrix}=a_0\begin{bmatrix}e_1\\ e_{k_1+1}\\ \vdots\\ e_{k_m+1}\\ \vdots\end{bmatrix}+(1-a_0^2)^{1/2}W\begin{bmatrix}\overline{te_0}\\ \overline{te_{k_1}}\\ \vdots\\ \overline{te_{k_m}}\\ \vdots\end{bmatrix}$$

it is easy to see that W is an unitary operator on the subspace $V=\mathrm{span}\{\overline{te_0}, \overline{te_{k_1}}, \cdots\overline{te_{k_m}}, \cdots\}$ Inner products with 1, we get

$$\begin{bmatrix}\langle\varphi+e_0, 1\rangle\\ \langle\varphi te_{k_1}, 1\rangle\\ \vdots\\ \langle\varphi te_{k_m}, 1\rangle\\ \vdots\end{bmatrix}=(1-a_0^2)^{1/2}W\begin{bmatrix}\overline{e_0(0)}\\ \overline{e_{k_1}(0)}\\ \vdots\\ \overline{e_{k_m}(0)}\\ \vdots\end{bmatrix}$$

since e_0 is outer, the left side cannot be a zero vector.

Lemma 5. With the same conditions as in Lemma 4, we have $a_n<1$ $\forall n\geq 0$

Proof. Assume $a_k=1$ for some k, by (1), we have

$$T_\varphi e_k=\varphi e_k=e_{k+1}$$

put $M=\{x\in H^2:\varphi x\in H^2\}$, then M is a non-trivial invariant subspace of U (the multiplicate operator M_t on H^2). Therefore, there exists an inner function ψ such that $$M=\psi H^2$$
So $\varphi=\bar{\psi}\chi$ where χ is an inner function, by [7], T_φ is a partial isometry, this is impossible except $a_n=1$ $\forall n\geq 0$ (which is a trivial case for the whole theorem).

Now we turn to prove our theorem, put $N=\{k: a_k=a_0\}$ N is a finite set, assume $N=\{k_0, k_1, k_2, \cdots k_m\}$ and $k_0=0$.
we are going to prove that N contains only one element.

Because

$$T_\varphi^* T_\varphi=\sum_{n=0}^{\infty}a_n^2 e_n\otimes e_n$$

$$U^*T_\varphi^*T_\varphi U=T_\varphi^*T_\varphi+U^*T_\varphi^*1\otimes U^*T_\varphi^*1$$

and $1=\sum_{n=0}^{\infty}\overline{e_n}(0)e_n \quad T_\varphi^* 1=\sum_{n=0}^{\infty}\overline{e_n(0)}a_{n-1}e_{n-1}=\sum_{n=0}^{\infty}\overline{e_{n+1}(0)}a_n e_n$

we have

$$\sum_{n=0}^{\infty}a_n^2 e_n\otimes e_n=\sum_{n=0}^{\infty}a_n^2 U^* e_n\otimes U^* e_n-\sum_{n=0}^{\infty}\langle ,U^* T_\varphi^* 1\rangle\overline{e_{n+1}(0)a_n}U^* e_n$$

$$=\sum_{n=0}^{\infty}[a_n^2\langle ,U^* e_n\rangle-\langle ,U^* T_\varphi^* 1\rangle\overline{e_{n+1}(0)}a_n]U^* e_n \qquad (2)$$

therefore $a_0^2 te_0=\sum_{n=0}^{\infty}[a_n^2\langle te_0,e_n\rangle-\langle te_0,T_\varphi^* 1\rangle\overline{e_{n+1}(0)}a_n]e_n$

$$-\sum_{n=0}^{\infty}[a_n^2\langle te_0,e_n\rangle-\langle te_0,T_\varphi^* 1\rangle\overline{e_{n+1}(0)}a_n]e_n(0)$$

inner products with e_0

$$a_0^2\langle te_0,e_0\rangle=a_0^2\langle te_0,e_0\rangle-\langle\varphi te_0,1\rangle\overline{e_1(0)a_0}$$

$$-\{\sum_{n=0}^{\infty}[a_n^2\langle te_0,e_n\rangle-\langle\varphi te_0,1\rangle\overline{e_{n+1}(0)}a_n]e_n(0)\}\overline{e_0(0)}$$

we have

$$a_0^2+e_0=\sum_{n=0}^{\infty}[a_n^2\langle te_0,e_n\rangle-\langle\varphi te_0,1\rangle\overline{e_{n+1}(0)}a_n]e_n+\frac{1}{e_0(0)}\langle\varphi te_0,1\rangle a_0\overline{e_1(0)} \qquad (3)$$

If $T_\varphi e_{k_i}=a_0 e_{k_i+1}$ $i=0,1,\cdots,m$. Inner products e_{k_i} with (3), we get

$$a_0^2\langle te_0,e_{k_i}\rangle=a_0^2\langle te_0,e_{k_i}\rangle-a_0\langle\varphi te_0,1\rangle\overline{e_{k_i+1}(0)}+\frac{1}{e_0(0)}\langle\varphi te_0,1\rangle a_0\overline{e_1(0)}\,\overline{e_{k_i}(0)}$$

$$\langle\varphi te_0,1\rangle\overline{e_{k_i+1}(0)}\,\overline{e_0(0)}=\langle\varphi te_0,1\rangle\overline{e_{k_i}(0)}\,\overline{e_1(0)}$$

by Lemma 4, $\langle\varphi te_{k_i},1\rangle$ $i=0,1,2,\cdots,m$ can not be all zeros. Without loss of generality we assume $\langle\varphi te_0,1\rangle\neq 0$ then we have $e_0(0)e_{k_i+1}(0)=e_1(0)e_{k_i}(0)$ it follows that

i) if $e_1(0)=0$, then $e_{k_i+1}(0)=0$, $i=0,1,2,\cdots m$.

ii) if $e_1(0)\neq 0$, then

$$e_{k_i+1}(0)=\frac{e_1(0)}{e_0(0)}e_{k_i}(0),\ i=0,1,2,\cdots m. \qquad (4)$$

Consider the subspace spaned by $\{e_0,e_{k_1},e_{k_2},\cdots e_{k_m}\}$ choose λ_1 and λ_2 such that

$$\lambda_1 e_{k_m}(0)+\lambda_2 e_{k_{m-1}}(0)=0 \qquad (5)$$

so

$$\frac{\lambda_1 e_{k_m}+\lambda_2 e_{k_{m-1}}}{t}\in H^2$$

we have

$$T_\varphi^* T_\varphi\frac{\lambda_1 e_{k_m}+\lambda_2 e_{k_{m-1}}}{t}=a_0 T_\varphi^* U^*(\lambda_1 e_{k_{m+1}}+\lambda_2 e_{k_{m-1}+1})$$

by (5) we have

$$U^*(\lambda_1 e_{k_m+1}+\lambda_2 e_{k_{m-1}+1})=\frac{\lambda_1 e_{k_m+1}+\lambda_2 e_{k_{m-1}+1}}{t}$$

therefore

$$T_{\varphi}^{*}T_{\varphi}\frac{\lambda_1 e_{k_m}+\lambda_2 e_{k_{m-1}}}{t}=a_0^2\frac{\lambda_1 e_{k_m}+\lambda_2 e_{k_{m-1}}}{t}$$

$$\frac{\lambda_1 e_{k_m}+\lambda_2 e_{k_{m-1}}}{t}\in\text{span }\{e_0, e_{k_1}, e_{k_2}, \cdots e_{k_m}\}$$

Now, use a similar process to that in [1], it is easy to show that

$$U^{*}\text{span }\{e_{k_i-1}, i=1,2,\cdots m\}\subset\text{span }\{e_{k_i-1} i=1,2,\cdots m\}$$

If $k_1\neq 1$, then $e_0\in\{\text{span }\{e_{k_i-1} i=1,2,\cdots,m\}\}^{\perp}$, but the later is an invariant subspace of U, since e_0 is outer, this is impossible [6].

If $k_1=1$.

Put $V=\{\text{span}\{e_{k_i-1}, i=1,2,\cdots,m\}\}^{\perp}$, there exists an inner function Ψ such taht,

$$V=\Psi H^2,$$

i.e.

$$\langle e_{k_i-1}, \psi H^2\rangle=0,\quad i=1,2,\cdots,m$$

It is easy to see that ψ must be with finite Blaschke product. Assume the zeros are $a_1, a_2, \cdots, a_m$ (in fact, the Blaschke product parts of ψ must be of order m), there exists $\beta_1, \beta_2, \cdots, \beta_m$ s.t.

$$e_0=\sum_{i=1}^{m}\frac{\beta_i}{1-\bar{a}_i t}$$

$$=\frac{\beta_1(1-\bar{a}_2 t)\cdots(1-\bar{a}_m t)+\cdots+\beta_m(1-\bar{a}_1 t)\cdots(1-\bar{a}_{m-1}t}{(1-\bar{a}_1 t)(1-\bar{a}_2 t)\cdots(1-\bar{a}_m t)}$$

$$=\frac{P(t)}{(1-\bar{a}_1 t)(1-\bar{a}_2 t)\cdots(1-\bar{a}_m t)}$$

By Lemma 2.

$$\varphi=r\frac{t e_0}{\bar{e}_0}$$

$$=r\cdot t\cdot\frac{P(t)}{\overline{P(t)}}\cdot\frac{(t-a_1)(t-a_2)\cdots(t-a_m)}{(1-\bar{a}_1 t)(1-\bar{a}_2 t)\cdots(1-\bar{a}_m t)}$$

let

$$P(t)=A_0(t-A_1)(t-A_2)\cdots(t-A_{m-1})$$

since e_0 is outer, $|A_j|\geqslant 1$, $j=1,2,\cdots,m-1$.

$$\frac{P(t)}{\overline{P(t)}}=\frac{A_0(t-A_1)(t-A_2)\cdots(t-A_{m-1})}{\bar{A}_0(\bar{t}-\bar{A}_1)(\bar{t}-\bar{A}_2)\cdots(\bar{t}-\bar{A}_{m-1})}$$

If for some j, $|A_j|>1$, then

$$\frac{t-A_j}{\bar{t}-\bar{A}_j}=\frac{A_j(\frac{1}{A_j}t-1)}{\bar{t}\bar{A}_j(\frac{1}{\bar{A}_j}-t)}=t\frac{A_j}{\bar{A}_j}\left(\frac{1-\frac{1}{A_j}t}{t-\frac{1}{\bar{A}_j}}\right)$$

If for some j, $|A_j|=1$, then

$$\frac{t-A_j}{\bar{t}-\bar{A}_j}=\frac{A_j}{\bar{t}}\left(\frac{\bar{A}_j t-1}{1-\bar{A}_j t}\right)=-tA_j$$

So $\frac{\overline{P(t)}}{P(t)} = t^l \cdot B \cdot S$ where B and $\overline{S}$ are finite Blaschke products and $0 \leq l \leq m-1$.

Therefore $\Phi = rt^{l+1} B \cdot S \cdot \frac{(t-a_1)(t-a_2)\cdots(t-a_m)}{(1-\overline{a}_1 t)(1-\overline{a}_2 t)\cdots(1-\overline{a}_m t)} = Q \cdot R$

where Q and $\overline{R}$ are finite Blaschke products. By [7], T_φ must be a partial isometry, by Lemma 5, this is impossible except the trivial case.

We can only have

$$a_0 \neq a_n \quad n = 1, 2, \cdots.$$

Thus completes the proof.

Corollary 1. If the weight sequence $\{a_n\}_{n=0}^{\infty}$ of the shift T is convergent ant $T \cong T_\varphi$ then

$$1 - a_n^2 = (1 - a_0^2)^{n+1} \qquad \forall n \geq 0$$

Proof. Without loss of generality, we assume $\lim_{n\to\infty} a_n = 1$. By Lemma 5, the set $N = \{k : a_k = a_0\}$ must be finite and by [8] we have $|\varphi| = 1$ a.e. So we get the conclusion.

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