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On the Generalized Topological Degree for Perturbed Maximal Monotone Mappings*

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In recent years, F.E. Browder has constructed the topological degree for nonlinear monotone mappings [2],[3]. In the paper [1], we have introduced the classes of mappings of type $(S)_+^*$ and of type quasi-$(S)_+^*$ and the concept of weakly-demicontinuity and we have constructed the generalized topological degree for these classes of mappings. Some important results in [2] and [3] have been improved in [1]. But, many problems in partial differential equations and integral equations may be described in term of seeking a solution of an abstract equation

$$g\in Tx+fx \qquad (1)$$

where T is a maximal monotone mapping of Banach space X to 2^{X^*}, f a mapping of X to its dual space X and $g\in X^*$ is given. For reason given above, first author of this paper and Browder have researched independently the topological degree for perturbed maximal monotone mappings $T+f$ [2],[3],[5]. However, it is assumed acctually that the space X is separable in [5], and the mapping f is considered only to be demicontinuous and bounded pseudomonotone in [2] and [3]. In this paper, we shall continue the work in [1]. First, we shall construct the generalized topological degree for the sum $T+f$, where T is maximal monotone and $f:\overline{\Omega}\subset X\to X^*$ a finitely continuous and quasibounded mapping of type quasi-$(S)_+^*$. Theorem 2 in [1] has shown that the mappings of type quasi-$(S)_+^*$ include of type quasibounded pseudomonotone, quasibounded generalized pseudomonotone, weakly-demicontinuous $(S)_+^*$ and of type (P). The mappings f considered by us is much broader than that f in [2] and [3]. Our results extend and improve the degree theory constructed in [2]—[5]. We eliminate the restriction of separability of space in [5]. Second, we obtain some new results for surjectivity and solvability of the equation (1) by using our degree theory.

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§ 1 The construction of degree function for perturbed maximal monotone mappings

Throughout this paper, X denotes a reflexive Banach space, X^* its dual space and "$\to$" and "$\rightharpoonup$" denote strong and weak convergence, respectively.

In this section, we shall consider the topological degree for the sum $T+f$, where $T:X\to 2^{X^*}$ is a maximal monotone mapping and $f:\overline{\Omega}\to X^*$ a finitly continuous and quasibounded mapping of type quasi-$(S)_+^*$. First, we constructed the topological degree for $T+f$ when f is a mapping of type $(S)_+^*$.

Definition 1 ([2]). Let $\{T_t\}_{t\in[0,1]}$ be a family of mappings from X to 2^{X^*}. $\{T_t\}_{t\in[0,1]}$ is said to be a pseudomonotone homotopy, if for any $\{t_n\}\subset[0,1]$, $t_n\to t$, $[u_n, w_n]\in G(T_n)$, $u_n\rightharpoonup u$, $w_n\rightharpoonup w$ and $\overline{\lim}(w_n, u_n)\leqslant(w, u)$, we have $w\in T_t(u)$ and $(w_n, u_n)\to(w, u)$.

Lemma 1 ([2]). Let $\{T_t\}_{t\in[0,1]}$ be a pseudomonotone homotopy of maximal monotone mappings from X to 2^{X^*} with $0\in T_t(0)$ for all $t\in[0,1]$. Suppose $\{u_n\}\subset X$, $u_n\rightharpoonup u$, $\{t_n\}\subset[0,1]$, $t_n\to t$, $\varepsilon_n>0$, $\delta_n>0$, $\varepsilon_n\to 0$, $\delta_n\to 0$, Let $V_n=T_{t_n,\varepsilon_n}(u_n)$, $Z_n=T_{t_n,\delta_n}(u_n)$. where $T_{t,\varepsilon}=(T_t^{-1}+\varepsilon J^{-1})^{-1}$ (J is the normalized duality mapping of X, so is in what following). If there exists $\{s_n\}\subset[0,1]$ such that $w_n=(1-s_n)V_n+s_nZ_n$, $w_n\rightharpoonup w\in X^*$ and $\overline{\lim}(w_n, u_n)<(w,u)$, we have $w\in T_t(u)$ and $(w_n, u_n)\to(w, u)$.

The following lemma can be proved easily by the definitions given in the paper [1].

Lemma 2. Let T_1 and T_2 be maximal monotone mappings from X to X^* with $T_1(0)=T_2(0)=0$. $D(T_1)=D(T_2)=X$. $\Omega\subset X$ a bounded open set, and $f:\overline{\Omega}\to X^*$ a weakly-demicontinuous mapping of type $(S)_+^*$. Then $\{f+tT_1+(1-t)T_2\}_{t\in[0,1]}$ is a homotopy of type $(S)_+^*$.

Theorem 1. Let $T:X\to 2^{X^*}$ be a maximal monotone mapping with $0\in T(0)$, $\Omega\subset X$ a bounded open set, and $f:\overline{\Omega}\to X^*$ a finitely continuous and weakly-demicontinuous quasibounded mapping of type $(S)_+^*$ For each $\varepsilon>0$, let $T_\varepsilon=(T^{-1}+\varepsilon J^{-1})^{-1}$ (Yosida approximant of T). Then

(i) If $y_0\in X^*$, $y_0\bar{\in}(T+f)(\partial\Omega)$, then there exist $\varepsilon_0>0$ such that for $0<\varepsilon$, $\delta<\varepsilon_0$, $t\in[0,1]$. we have $y_0\bar{\in}(t(T_\delta+f)+(1-t)(T_\varepsilon+f))(\partial\Omega)$;

(ii) For each $\varepsilon>0$, $T_\varepsilon+f:\overline{\Omega}\to X^*$ is a weakly-demicontinuous mapping of type $(S)_+^*$. Hence, $\deg(T_\varepsilon+f, \Omega, y_0)$ is well-defined for $0<\varepsilon<\varepsilon_0$. Moreover, $\deg(T_\varepsilon+f,\Omega, y_0)$ is independent of $\varepsilon\in(0,\varepsilon_0)$.

Proof. (i) If the conclusion is false, then there exist $\{u_n\}\subset\partial\Omega$, $\varepsilon_n\to 0$, $\delta_n\to 0$ and $\{t_n\}\subset[0,1]$ such that $u_n\rightharpoonup u$ and $t_nT_{\delta_n}(u_n)+(1-t_n)T_{\varepsilon_n}(u_n)+f(u_n)=y_0$. By $0\in T(0)$, we have $(t_nT_{\delta_n}(u_n)+(1-t_n)T_{\varepsilon_n}(u_n), u_n)\leqslant 0$. Thus, $(f(u_n), u_n)\leqslant(y_0, u_n)\leqslant\|y_0\|\|u_n\|$ Since f is quasibounded, there exists a subsequence $\{f(u_{n_k})\}$ of $\{f(u_n)\}$

such that $f(u_{n_k}) \rightharpoonup g \in X^*$. Hence, $t_{n_k} T\delta_{n_k}(u_{n_k}) + (1-t_{n_k}) T\varepsilon_{n_k}(u_{n_k}) \rightharpoonup y_0 - g$. By proposition 2 in the paper [1], we have $\underline{\lim}(f(u_n), u_n - u) \geq 0$. So

$$\overline{\lim}(t_{n_k} T\delta_{n_k}(u_{n_k}) + (1-t_{n_k}) T\varepsilon_{n_k}(u_{n_k}), u_{n_k} - u)$$
$$= \overline{\lim}(y_0 - f(u_{n_k}), u_{n_k} - u) = -\underline{\lim}(f(u_{n_k}), u_{n_k} - u) \leqslant 0$$

$\{T_t = T\}_{t \in [0,1]}$ is a pseudomonotone homotopy in the sense of Definition 1. By Lemma 1, $y_0 - g \in T(u)$, $\lim(t_{n_k} T\delta_{n_k}(u_{n_k}) + (1-t_{n_k}) T\varepsilon_{n_k}(u_{n_k}), u_{n_k} - u) = 0$. Hence, $\lim(f(u_{n_k}), u_{n_k} - u) = 0$. Because f is a weakly-demicontinuous mapping of type $(S)_+^*$, we have $u_{n_k} \to u \in \partial\Omega$, $f(u_{n_k}) \rightharpoonup f(u)$. It follows that $y_0 \in (T+f)(u)$, which contradicts $y_0 \bar{\in} (T+f)(\partial\Omega)$.

(ii) By Lemma 2. $T_\varepsilon + f: \overline{\Omega} \to X^*$ is a weakly-demicontinuous mapping of type $(S)_+^*$ for each $\varepsilon > 0$, and

$$\{t(T_\delta + f - y_0) + (1-t)(T_\varepsilon + f - y_0)\}_{t \in [0,1]} = \{tT_\delta + (1-t)T_\varepsilon + f - y_0\}_{t \in [0,1]}$$

is a homotopy of type $(S)_+^*$. By the conclusion in (i), for $0 < \varepsilon, \delta < \varepsilon_0$, $t \in [0,1]$, $0 \bar{\in} [t(T_\delta + f - y_0) + (1-t)(T_\varepsilon + f - y_0)](\partial\Omega)$. From Theorem 6 in the paper [1], we have

$$\deg(T_\varepsilon + f - y_0, \Omega, 0) = \deg(T_\delta + f - y_0, \Omega, 0)$$

i.e., $\deg(T_\varepsilon + f, \Omega, y_0) = \deg(T_\delta + f, \Omega, 0)$.

So $\deg(T_\varepsilon + f, \Omega, y_0)$ is independent of ε in $(0, \varepsilon_0)$. Q.E.D.

Definition 2. Let $T: X \to 2^{X^*}$ be a maximal monotone mapping with $0 \in T(0)$, $G \subset X$ a bounded open set, and $f: \overline{\Omega} \to X^*$ a finitely continuous and weakly-demicontinuous quasibounded mapping of type $(S)_+^*$, $T_\varepsilon = (T^{-1} + \varepsilon J^{-1})^{-1}$, $y_0 \bar{\in} (T+f)(\partial\Omega)$. we define

$$\deg(T + f, \Omega, y_0) = \lim_{\varepsilon \to 0} \deg(T_\varepsilon + f, \Omega, y_0).$$

Remark 1. We have constructed the topological degree for the sum $T+f$ by using Yosida approximation. Because we demand that f is quasibounded, the degree theory for the sum $T+f$ can not include the topological degree constructed for the mappings of type $(S)_+^*$ in the paper [1].

In the following, we shall give the some properties on the topological degree of mappings of type $T+f$.

Theorem 2 (Solvability). Let $T: X \to 2^{X^*}$ be a maximal monotone mapping with $0 \in T(0)$, $\Omega \subset X$ a bounded open set, and $f: \overline{\Omega} \to X^*$ a finitely continuous and weakly-demicontinuous quasiboundgd mapping of type $(S)_+^*$. If $\deg(T+f, \Omega, y_0) \neq 0$. then $y_0 \in (T+f)(\Omega)$.

Proof. By Definition 2, there exists $\varepsilon_0 > 0$ such that $\deg(T_\varepsilon + f, \Omega, y_0) \neq 0$ $(\varepsilon \in (0, \varepsilon_0))$. Let $\varepsilon_n \to 0$, by Theorem 4 in the paper [1], there exists $\{u_n\} \subset \Omega$ such that $u_n \rightharpoonup u$ and $T\varepsilon_n(u_n) + f(u_n) = y_0$. Because $(T\varepsilon_n(u_n), u_n) \geqslant 0$ and f is quasibounded, we have a subsequence $\{f(u_{n_k})\}$ of $\{f(u_n)\}$ such that $f(u_{n_k} \rightharpoonup g$. Thus, $T\varepsilon_{n_k}(u_{n_k}) \rightharpoonup$

$w=y_0-g$. Since

$$\overline{\lim}\,(T\varepsilon_n(u_n),u_n-u)=\overline{\lim}\,(y_0-f(u_n),u_n-u)=-\underline{\lim}\,(f(u_n),u_n-u)\leqslant 0,$$

it follows $\overline{\lim}\,(T\varepsilon_{n_k}(u_{n_k}),u_{n_k}-u)\leqslant\overline{\lim}\,(T\varepsilon_n(u_n),u_n-u)\leqslant 0$.

By Lemma 1, $w\in T(u)$, $\lim(T\varepsilon_{n_k}(u_{n_k}),u_{n_k}-u)=0$. So $\lim(f(u_{n_k}),u_{n_k}-u)=0$. it follows $u_{n_k}\to u\in\overline{\Omega}$ and $f(u_{n_k})\rightharpoonup f(u)$. Hence, $y_0=f(u)+w\in(T+f)(\Omega)\subset(T+f)(\overline{\Omega})$. By the condition $y_0\bar{\in}(T+f)(\partial\Omega)$. we have $y_0\in(T+f)(\Omega)$. Q.E.D.

In order to construct the homotopy invariance, we make the following preparation.

Definition 3. Let $\{f_t\}_{t\in[0,1]}$ be a family of the mappings from X to X^*. $\{f_t\}_{t\in[0,1]}$ is said to be unifom quasibounded. if for each $M>0$, there exists a $C>0$ such that for arbitrary $t\in[0,1]$, when $x\in D(f_t)$ and $(f_t(x),x)\leqslant M\|x\|$, $\|x\|\leqslant M$. we have $\|f_t(x)\|\leqslant C$.

Proposition 1. Let $f:X\to X^*$ be a quasibounded mapping. Then $\{tf+(1-t)J\}_{t\in[0,1]}$ is a family of the uniform quasibounded mappings.

Proof. Let $M>0$. Because f is quasiboundgd, there exists a $C_1>0$ such that for $x\in D(f)$, $(f(x),x)\leqslant M\|x\|$ and $\|x\|\leqslant M$, we have $\|f(x)\|\leqslant C_1$. Let $C=\max\{M,C_1\}$. For $t\in(0,1]$, $x\in D(f)$ and $(tf(x)+(1-t)Jx,x)\leqslant tM\|x\|$, we have $(f(x),x)\leqslant M\|x\|$. Hence, $\|f(x)\|\leqslant C_1\leqslant C$. By $\|Jx\|=\|x\|\leqslant M\leqslant C$, we obtain $\|tf(x)+(1-t)Jx\|\leqslant C$. For $t=0$, $\|tf(x)+(1-t)Jx\|=\|Jx\|\leqslant C$. So $\{tf+(1-t)J\}_{t\in[0,1]}$ is uniform quasibonded. Q,E.D.

Lamma 3. Let $\{T_t\}_{t\in[0,1]}$ be a pseudomonotone homotopy of maximal monotone mappings from X to 2^{X^*}, and $f:X\to X^*$ a bounded maximal monotone mapping with $D(f)=X$. Then $\{T_t+f\}_{t\in[0,1]}$ is a pseudomonotone homotopy.

Proof. Let $\{t_n\}\subset[0,1]$, $t_n\to t$, $[u_n,\omega_n]\in G(Tt_n)$, $u_n\rightharpoonup u$, $\omega_n+f(u_n)\rightharpoonup g$, and $\overline{\lim}\,(\omega_n+f(u_n),u_n)\leqslant(g,u)$. Because f is bounded, there exists a subsequence $\{f(u_{n_k})\}$ of $\{f(u_n)\}$ such that $f(u_{n_k})\rightharpoonup g_1$. Thus, $\omega_{n_k}\rightharpoonup g-g_1=\omega$. By $\underline{\lim}\,(f(u_{n_k}),u_{n_k}-u)\geqslant 0$, we have $\overline{\lim}\,(\omega_{n_k},u_{n_k})\leqslant(\omega,u)$. From Definition, $\omega\in T_{t(u)}$, $\lim(\omega_{n_k},u_{n_k}-u)=0$. So $\overline{\lim}\,(f(u_{n_k}),u_{n_k}-u)\leqslant 0$. Since f is maximal monotone mapping, we have $g_1=f(u)$ and $\lim(f(u_{n_k}),u_{n_k}-u)=0$. Thus, $g=f(u)+\omega\in(Tt+f)(u)$, $\lim(\omega_{n_k}+f(u_{n_k}),u_{n_k})=(g,u)$. It follows $\lim(\Omega_n+f(u_n),u_n)=(g,u)$. By Definition 1, we know that the conclusion is true. Q.E.D.

Lemma 4. Let $\{T_t\}_{t\in[0,1]}$ be a family of the uniform bounded maximal monotone mappings from X to X^* with $0\in T_t(0)$ and $D(T_t)=X$ for all $t\in[0,1]$, and $\{T_t\}_{t\in[0,1]}$ a pseudomonotone homotopy, $\Omega\subset X$ a buunded open set, $\{f_t\}_{t\in[0,1]}$ a homotopy of class $(S)_+^*$ from $\overline{\Omega}$ to X^*. Then $\{T_t+f_t\}_{t\in[0,1]}$ is ao a homotopy of class $(S)_+^*$ from $\overline{\Omega}$ to X^*. class

Proof. Similar to the proof of Lemma 3, we omit it. Q.E.D.

Theorem 3 (Homotopy invariance). Let $\Omega\in X$ be a bounded open set.

$\{f_t\}_{t\in[0,1]}$ a family of uniform quasibounded and finitely conttnuus mappings, and $\{f_t\}_{t\in[0,1]}$ be a homotopy of class $(S)_+^*$ from $\overline{\Omega}$ to X^*, $\{T_t\}_{t\in[0,1]}$ a pseudomonotone homotopy of maximal monotone mappings from X to 2^{X^*} with $0\in T_t(0)$ and $0\overline{\in}(T_t+f_t)(\partial\Omega)$ for all $t\in[0,1]$. Then deg $(T_t+f_t,\Omega,0)$ is independent of $t\in[0,1]$.

Proof. (i) We prove that $\{T_{t,\varepsilon}+f_t\}_{t\in[0,1]}$ is a homotopy of class $(S)_+^*$ from $\overline{\Omega}$ to X^* for each $\varepsilon>0$.

By $\{T_t\}_{t\in[0,1]}$ is a pseudomonotone homotopy and Definition 1, we know that $\{T_t^{-1}\}_{t\in[0,1]}$ is also pseudomonotone homotopy. From, Lemma 3, $\{T_t^{-1}+\varepsilon T^{-1}\}_{t\in[0,1]}$ is a pseudomonotone homotopy. Thus, $\{T_{t,\varepsilon}\}_{t\in[0,1]}$ is also a pseudomonotone homotopy. Since $\{T_{t,\varepsilon}\}_{t\in[0,1]}$ is uniform bounded for each $\varepsilon>0$, we know that $\{T_{t,\varepsilon}+f_t\}_{t\in[0,1]}$ is a homotopy of class $(S)_+^*$ by Lamma 4.

(ii) We shall prove that there exists $\varepsilon_0>0$ such that $0\overline{\in}(T_{t,\varepsilon}+f_t)(\partial\Omega)$ for each $\varepsilon\in(0,\varepsilon_0)$ and deg $(T_{t,\varepsilon}+f_t,\Omega,0)$ is independent of $t\in[0,1]$ and $\varepsilon\in(0,\varepsilon_0$.

Suppose that the conclusion is false. By Lemma 2 and Theorem 4 in the paper [1], there exist $\{t_n\}\subset[0,1]$, $\varepsilon_n>0,\delta_n>0$, $t_n\to t$, $\delta_n\to 0$, $\varepsilon_n\to 0$, $u_n\in\partial\Omega$ and $\{s_n\}\subset[0,1]$ such that $(1-s_n)T_{t_n,\varepsilon_n}(u_n)+s_nT_{t_n,\delta_n}(u_n)+f_{t_n}(u_n)=0$ $(n=1,2,\cdots)$.

Let $\Omega_n=-f_{t_n}(u_n)$. Since $0\in T_{t,\varepsilon}(0)$ for $\varepsilon>0$ and $t\in[0,1]$, we have $(f_{t_n}(u_n),u_n)\leqslant 0$. By $\{f_t\}_{t\in[0,1]}$ is uniform quasibounded, we may assume $u_n\rightharpoonup u$ and $\omega_n\rightharpoonup\omega$. By Proposition 8 in the paper [1], $\overline{\lim}(\Omega_n,u_n-u)=-\underline{\lim}(f_{t_n}(u_n),u_n-u)\leqslant 0$. From Lemma 1, we have $\Omega\in T_t(u)$ and $\lim(\Omega_n,u_n-u)=0$. So $\lim(f_{t_n}(u_n),u_n-u)=0$. Because $\{f_t\}_{t\in[0,1]}$ is a homotopy of class $(S)_+^*$, it follows $u_n\to u$ and $f_{t_n}(u_n)\rightharpoonup f_t(u)$. Thus, $u\in\partial\Omega$ and $-f_t(u)=\Omega\in T_t(u)$. i.e. $0\in(T_t+f_t)(\partial\Omega)$. which contradicts $0\overline{\in}(T_t+f_t)(\partial\Omega)$. Q.E.D.

Remark 2. Because the additivity of domain is similar to the paper [1], we do not discuss it here.

In the following, we consider the generalized topological degree for $T+f$ when f is a mapping of type quasi-$(S)_+^*$.

Theorem 4. Let $\Omega\subset X$ be a bounded open set, $f:\overline{\Omega}\to X^*$ a finitely continuous and quasibounded mapping of type quasi-$(S)_+^*$, and $T:X\to 2^{X^*}$ a maximal monotone mapping with $0\in T(0)$, $y_0\overline{\in}\overline{(T+f)(\partial\Omega)}$. Then

(i) There exists $\varepsilon_0>0$ such that $y_0\overline{\in}(T+f+\varepsilon J)(\partial\Omega)$ for $\varepsilon\in(0,\varepsilon_0)$;

(ii) deg $(T+f+\varepsilon J,\Omega,y_0)$ is independent of $\varepsilon\in(0,\varepsilon_0)$.

Proof. (i) Because Ω and J is bounded, there exists $\varepsilon_0>0$ such that $y_0\overline{\in}(T+f+\varepsilon J)(\partial\Omega)$ for $\varepsilon\in(0,\varepsilon_0)$. Thus, deg $(T+f+\varepsilon J,\Omega,y_0)$ is well-defined.

(ii) For $0<\varepsilon<\varepsilon_0$, $0<\delta<\varepsilon_0$ and $t\in[0,1]$, we obtain

$$t(T+f+\varepsilon J)+(1-t)(T+f+\delta J)=T+f+(t\varepsilon+(1-t)\delta)J.$$

So
$$y_0\overline{\in}(t(T+f+\varepsilon J)+(1-t)(T+f+\delta J))(\partial\Omega).$$

i.e.
$$0\overline{\in}(T+(f-y_0)+(t\varepsilon+(1-t)\delta)J)(\partial\Omega).$$

By Lamma 2 and Theorem 3, we have

$$\deg(T+f+\varepsilon J-y_0,\Omega,0)=\deg(T+f+\delta J-y_0,\Omega,0).$$

Hence, $\deg(T+f+\varepsilon J,\Omega,y_0)=\deg(T+f+\delta J,\Omega,y_0)$ Q.E.D.

Definition 4. Let $\Omega\subset X$ be a bounded open set, $f:\overline{\Omega}\to X^*$ a finitely continuous and quasibounded mapping of type quasi-$(S)_+^*$, and $T:X\to 2^{X^*}$ a maximal monotone mapping with $0\in T(0)$, $y_0\overline{\in}\overline{(T+f)(\partial\Omega)}(\partial\Omega)$. We define

$$\deg(T+f,\Omega,y_0)=\lim_{\varepsilon\to 0}\deg(T+f+\varepsilon J,\Omega,y_0).$$

In the following, we give two properties for the degree function. The proofs are omitted.

Theorem 5 (Solvability) Let $\Omega\subset X$ be a bounded open set, $f:\overline{\Omega}\to X^*$ a finitely continuous and quasibounded mapping of type quasi-$(S)_+^*$, and $T:X\to 2^{X^*}$ a maximal monotone mapping with $0\in T(0)$. If $\deg(T+f,\Omega,y_0)\neq 0$, then $y_0\in\overline{(T+f)(\Omega)}$.

Theorem 6 (Homotopy invariance) Let $\{T_t\}_{t\in[0,1]}$ be a pseudomonotone homotopy of maximal monotone mappings from X to 2^{X^*} with $0\in T_t(0)$ $(t\in[0,1])$, $\Omega\subset X$ a bounded open set, and $\{f_t\}_{t\in[0,1]}$ a homo topy of class quasi-$(S)_+^*$ from $\overline{\Omega}$ to X^* and uniform quasibouded, there exists $r>0$ such that $B(0,r)\cap(T_t+f_t)(\partial\Omega)=\phi$ $(t\in[0,1])$ Then $\deg(T_t+f_t,\Omega,0)$ is independent of $t\in[0,1]$.

Remark 3. The topological degree for $T+f$ had been constructed in the paper [2] and [3] when f is a bounded and demicontinuous pseudomonotone mapping. In this section, we extend f to a finitely continuous and quasibounded mapping of type quasi-$(S)_+^*$. Because zero mapping is of type quasi-$(S)_+^*$. the topological degree for $T+f$ is the topological degree for T when $f=0$. where T is a maximal monotone mapping. The topologcial degree for T had been constructed in the paper [4] when X is a separable Hilbert space and int $D(T)\neq\phi$. Hence, we also extend the degree theory constructed in the paper [4].

§ 2 Solvability and Surjectivity

In this section, we shall prove some results for solvability and surjectivity by using the degree theory constructed in the paper [1] and §1.

Definition 5. Let $f:X\to X^*$. f is called to satisfy condition (Q), if for $\{x_n\}\subset D(f)$, $x_n\to x_0\in D(f)$ and $f(x_n)\to y_0$, we have $f(x_0)=y_0$.

Lemma 5. Let $\Omega\subset X$ be a bounded convex set, and $f:\overline{\Omega}\to X^*$ satisfy condition (Q). Then $\overline{f(\Omega)}\subset f(\overline{\Omega})$.

Proof. Let $y_0\in\overline{f(\Omega)}$. Since Ω is a bounded convex set, we have $\{x_n\}\subset\Omega$ such that $x_n\to x\in\overline{\Omega}$, $f(x_n)\to y_0$. By condition (Q), $y_0=f(x)$. Hence $y_0\in f(\overline{\Omega})$.

Q.E.D.

Theorem 7. Let $\Omega\subset X$ a bounded open convex set, and $f:\overline{\Omega}\to X^*$ a finitely continuous mapping of type quasi-$(S)_+^*$. For $g\in X^*$. there exist $r>0$ and $c>\|g\|$ such that $B(0,r)\subset\overline{\Omega}$. $(f(x),x)\geqslant c\|x\|$ $(x\in\partial B(0,r))$. Then

(i) $g\in\overline{f(B(0,r))}$;

(ii) When f satisfies condition (Q), the equation $f(x)=g$ has at least a solution in $B(0,r)$.

Proof. Let $f_t(x)=t(f(x)-g)+(1-t)Jx,\ t\in[0,1],\ x\in\overline{B(0,r)}$. By Theorem 9 in the paper [1], $\{f_t\}_{t\in[0,1]}$ is a homotopy of type quasi-$(S)_+^*$. For $x\in\partial B(0,r)$.

$$(f_t(x),x)=t(f(x)-g,x)+(1-t)(Jx,x)\geqslant\min\{\ \ -\|g\|,\ r\}\|x\|.$$

So $B(0,S)\cap f_t(\partial B(0,r))=\phi,\ t\in[0,1],\ S=\frac{1}{2}\min\{c-\|g\|,r\}>0$, By Theorem 10 in the paper [1].

$$\deg(f-g,B(0,r),0)=\deg(J,B(0,r),0)=1.$$

Thus, $\deg(f,B(0,r),g)\neq0$. By Theorem 8 in the paper [1], we have $g\in\overline{f(B(0,r))}$.

If f satisfies condition (Q). By Lemma 5 and $g\overline{\in} f(\partial B(0,r))$, we have $g\in f(B(0,r))$. i.e. $f(x)=g$ has at least a solution in $B(0,r)$.

Corollary 1. Let $\Omega\subset X$ be a bounded open convex set, and $f:\overline{\Omega}\to X^*$ a finitely continuous mapping. For $g\in X^*$, there exist $r>0$ and $c>\|g\|$ such that $B(0,r)\subset\overline{\Omega}$, $(f(x),x)\geqslant c\|x\|$ $(x\in\partial B(0,r))$. If f satisfies one of the following conditions:

(i) f is a quasibounded generalized pseudomonotone mapping,

(ii) f is a demicontinuous pseudomonotone mapping, then the equation $f(x)=g$ has at least a solution in $B(0,r)$.

Proof. By Theorem 2 in the paper [1], we have that f is a mapping of type quasi-$(S)_+^*$. We can prove easily that f satisfies the condition (Q). From Theorem 7, we know that the conclusion is true. Q.E.D.

We can obtain the following result from Theorem 7.

Theorem 8 Let $f:X\to X^*$ be a finitely continuous and coercive mapping of type quasi-$(S)_+^*$, $D(f)=X$. Then

(i) $\overline{R(f)}=X^*$.

(ii) When f satisfies the condition (Q), $R(f)=X^*$.

Corollary 2. Let $f:X\to X^*$ be a demicontinuous and coercive mapping of type (P) with $D(f)=X$. Then $\overline{R(f)}=X^*$.

Corollary 3. Let $f:X\to X^*$ be a coercive mapping with $D(f)=X$. If f satisfies one of the following conditions:

(i) f is a finitely continuous and quasibounded generalized pseudomonotone mapping;

(ii) f is a demicontinuous pseudomonotone mapping, Then $R(f)=X^*$.

In the following, we shall consider the surjectivity for the sum of two mappings.

Definition 6. Let $f:X\to X^*$. f is called to satisfy condition (G), if for $\{x_n\}\subset D(f)$, $x_n\to x\in D(f)$, $f(x_n)\to g\in X^*$ and $\lim(f(x_n), x_n-x)=0$, we have $f(x)=g$.

By Definition 5 and Definition 6, we know that the condition (Q) is weaker than the condition (G).

Lemma 6. Let $T:X\to 2^{X^*}$ be a maximal monotone mapping with $0\in D(T)$, $\Omega\subset X$ a bounded convex set, and $f:\overline{\Omega}\to X^*$ a quasibounded mapping of type quasi-$(S)_+^*$ with the condition (G). Then $\overline{(T+f)(\Omega)}\subset(T+f)(\overline{\Omega})$.

Proof. Let $y_0\in\overline{(T+f)(\Omega)}$. Since Ω is a bounded convex set, there exists $\{x_n\}\subset\Omega$ such that $x_n\rightharpoonup x\in\overline{\Omega}$ and $g_n+f(x_n)\to y_0$, $y_n\in T(x_n)$. Thus, $\lim(g_n+f(x_n), x_n-x)=0$ and there exists $M>0$ such that $(g_n+f(x_n), x_n)\leqslant M\|x_n\|$ $(n=1,2,\cdots)$. Let $g_0\in T(0)$, then $(f(x_n), x_n)\leqslant M\|x_n\|+\|g_0\|\|x_n\|=(M+\|g_0\|)\|x_n\|$ $(n=1,2,\cdots)$.
By Theorem 1 in the paper [1], $\lim(f(x_n), x_n-x)\geqslant 0$. Since $\underline{\lim}(g_n, x_n-x)\geqslant 0$, we have $\lim(f(x_n), x_n-x)=0$ and $\lim(g_n, x_n-x)=0$. Because f is quasibounded, there exists a subsequence $\{f(x_{n_k})\}$ of $\{f(x_n)\}$ such that $f(x_{n_k})\rightharpoonup g\in X^*$. By the condilion (G), we have $g=f(x)$. Hence, $g_{n_k}\rightharpoonup y_0-f(x)$. it follows $y_0-f(x)\in Tx$. i.e. $y_0\in(T+f)(x)\subset(T+f)(\overline{\Omega})$. Q.E.D.

Theorem 9. Let $T:X\to 2^{X^*}$ be a maximal monotone mapping with $0\in D(T)$, $\Omega\subset X$ a bounded open convex set, and $f:\overline{\Omega}\to X^*$ a finitely continuous and quasibounded mapping of type quasi-$(S)_+^*$ with the condition (G). For $g\in X^*$ and $g_0\in T(0)$. If there exist $r>0$ and $c>\|g\|+\|g_0\|$ such that $B(0,r)\subset\overline{\Omega}$. $(f(x),x)\geqslant c\|x\|$ $(x\in\partial B(0,r))$. Then the equation $g\in(T+f)(x)$ has at least a solution in $B(0,r)$.

Proof. We write $T_1=T-y_0$, then T_1 is a maximal monotone mapping with $0\in T_1(0)$. Let $\overline{T}_\lambda=(T_1^{-1}+\lambda J^{-1})^{-1}$,

$$f_t(x)=t(\overline{T}_\lambda(x)+f(x)-g+g_0+\varepsilon Jx)+(1-t)Jx,\quad x\in\overline{B(0,r)},\quad \varepsilon>0.$$

By Theorem 5 in the paper [1], $\{f_t\}_{t\in[0,1]}$ is a homotopy of class $(S)_+^*$ for each $\varepsilon>0$. For $x\in\partial B(0,r)$,

$$\begin{aligned}(f_t(x),x)&=t(\overline{T}_\lambda(x)+f(x)-g+g_0+\varepsilon Jx, x)+(1-t)(Jx,x)\\ &\geqslant t(f(x)-g+g_0,x)+(1-t)(Jx,x)\\ &\geqslant\min\{c-\|g\|-\|g_0\|, r\}\|x\|.\end{aligned}$$

So $B(0,s)\cap f_t(\partial B(0,r))=\phi$, where $S=\frac{1}{2}\min\{c-\|g\|-\|g_0\|, r\}$. By Theorem 6 in the paper [1].

$$\deg(\overline{T}_\lambda+f+\varepsilon J-g+g_0, B(0,r),0)=\deg(T,B(0,r),0)=1.$$

i.e. $\deg(\overline{T}_\lambda+f+\varepsilon J, B(0,r), g-g_0)=1,\quad \lambda>0,\ \varepsilon>0.$

Hence $\deg(T_1+f+\varepsilon J, B(0,r), g-g_0)=\lim_{\lambda\to 0}\deg(\overline{T}_\lambda+f+\varepsilon J, B(0,r), g-g_0)=1.$

$$\deg(T_1+f, B(0,r), g-g_0)=\lim_{\varepsilon\to 0}\deg(T_1+f+\varepsilon J, B(0,r), g-g_0)=1.$$

By Theorem 5, $g-g_0\in\overline{(T+f)(B(0,r))}$. From Lemma 6 and $g-g_0\overline{\in}(T_1+f)(\partial B(0,r))$. we have $g-g_0\in(T_1+f)(B(0,r))$. Hence, $g\in(T+f)(B(0,r))$. Q.E.D.

Corollary 4. Let $T:X\to 2^{X^*}$ be a maximal monotone mapping with $0\in D(T)$, $\Omega\subset X$ a bounded oren convex set, and $f:\overline{\Omega}\to X^*$ a finitely continuous and quasi-bounded geperalized pseudomonotone mapping. Fro $g\in X^*$ and $g_0\in T(0)$, there exist $r>0$ and $c>\|g\|+\|g_0\|$ such that $B(0,r)\subset\overline{\Omega}$, $(f(x),x)\geqslant c\|x\|$ $(x\in\partial B(0,r))$. Then the equation $g\in(T+f)(x)$ has at least a solution in $B(0,r)$.

By Theorem 9, we can obtain the following result.

Theorem 10. Let $T:X\to 2^{X^*}$ be a maximal monotone mapping with $0\in D(T)$, and $f:X\to X^*$ a finitely continuous and quasibounded mapping of type quasi-$(S)_+^*$ with $D(f)=X$, f satisfy the condition (G) and be coercive. Then $R(T+f)=X^*$.

Corollary 5. Let $T:X\to 2^{X^*}$ be a maximal monotone mapping with $0\in D(T)$, and $f:X\to X^*$ a finitely continuous and quasibounded generalized pseudomonotone mapping with $D(f)=X$. If f is coercive, then $R(T+f)=X^*$.

Remark 4. The other proofs of Corollary 2, Corollary 3 and Corollary 5 can be found in the paper [8].

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