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Maximal Quotient Rings of Endomorphisms of Quasigenerators*

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0. Preliminaries. Let R be an associative ring with identity, and let Mod-R denote the category of all unital right R-modules. A set of right ideal $\mathcal{F}$ of R is called a Gabriel topology on R if $\mathcal{F}$ satisfies

T1. If $I\in\mathcal{F}$ and $I\subseteq J$, then $J\in\mathcal{F}$.

T2. If I and J belong to $\mathcal{F}$, then $I\cap J\in\mathcal{F}$.

T3. $I\in\mathcal{F}$ and $r\in R$, then $(I:r)=\{x\in R: rx\in I\}\in\mathcal{F}$.

T4. If I is a right ideal of R and there exists $J\in\mathcal{F}$ such that $(I:r)\in\mathcal{F}$ for every $r\in J$, then $I\in\mathcal{F}$.

The set of all essential right ideals of R forms a Gabriel topology, called the Goldie topology on R. If E is a injective right R-module, then the set $\mathcal{F}_E^0=\{I: I$ is right ideal of R, $\mathrm{Hom}_R(R/I,E)=0\}$ is a Gabriel topology on R, and it is called the Gabriel topology on R cogenerated by E. Specially when $E=E(R)$, the injective hull of R, $\mathcal{D}=\mathcal{F}_E^0$ is called the dense topology on R.

Associated with each Gabriel topology $\mathcal{F}$ on R, there exists a left exact torsion radical $t(M)$ of Mod-R such that $t(M)=\{x\in M: \mathrm{Ann}_R(x)\in\mathcal{F}\}, M\in$ Mod-R. For $M\in$ Mod-R, the quotient module $M_{\mathcal{F}}$ of M with respect to $\mathcal{F}$ is defined as

$$M_{\mathcal{F}}=\varinjlim \mathrm{Hom}_R(I,M/t(M)),\ I\in\mathcal{F}.$$

For other terminology about localization, the reader will refer to [1].

1. Let $\mathcal{F}$ be a Gabriel topology on R and let t be the associated torsion radical. For each right R-module M_R, the quotient module $M_{\mathcal{F}}$ of M can be defined as

$$M_{\mathcal{F}}=\varinjlim_{I\in\mathcal{F}}\mathrm{Hom}_R(I,M/t(M)).$$

In a similar way, if $P\in$ Mod-R and $\mathcal{F}(P)$ is the set of all $\mathcal{F}$-dense submodules of P, we can get an additive abelian group

$$P_{\mathcal{F}}\mathrm{Hom}_R(P,M)=\varinjlim_{P'\in\mathcal{F}(P)}\mathrm{Hom}_R(P',M/t(M)).$$

Define a pairing $P_{\mathcal{F}}\mathrm{Hom}_R(P,M)\times P_{\mathcal{F}}\mathrm{Hom}_R(P,P)\to P_{\mathcal{F}}\mathrm{Hom}_R(P,M)$ as follows: suppose $x\in P_{\mathcal{F}}\mathrm{Hom}_R(P,M)$ and $a\in P_{\mathcal{F}}\mathrm{Hom}_R(P,P)$ are represented by $\xi: P'\to M/T(M)$ and

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$a: P'' \to P/t(P)$; ξ induced a homomorphism $\bar{\xi}: P'+t(P)/t(P) \cong P'/t(P') \to M/t(M)$; we then define $xa \in P_{\mathscr{F}} \mathrm{Hom}_R(P, M)$ to be represented by the composed homomorphisms.

$$a^{-1}(P'+t(P)/t(P)) \xrightarrow{a} P'+t(P)/t(P) \xrightarrow{\bar{\xi}} M/t(M),$$

it is easy to see that xa is well defined; and the pairing is biadditive. When $P=M$, this makes $P_{\mathscr{F}} \mathrm{Hom}_R(P, P)$ a ring (briefly, we denote it by $P_{\mathscr{F}} \mathrm{End}_R P$), and in the general case it makes $P_{\mathscr{F}} \mathrm{Hom}_R(P, M)$ a right $P_{\mathscr{F}} \mathrm{End}_R P$-module, We call the elements in $P_{\mathscr{F}} \mathrm{Hom}_R(P, M)$ ($P_{\mathscr{F}} \mathrm{End}_R P$) partial homomorphisms (partial endomorphisms) from P to M (of P) with respect to $\mathscr{F}$.

Theorem 1.1. Let P_R, M_R be right R-modules, and let $\mathscr{F}$ be a Gabriel topology on R. Then there exists a additive group homomorphism

$$\Phi_{P,M}: P_{\mathscr{F}} \mathrm{Hom}_R(P, M) \to \mathrm{Hom}_{R_{\mathscr{F}}}(P_{\mathscr{F}}, M_{\mathscr{F}})$$

such that

(i) $\Phi_{P,P}$ is a ring isomorphism.

(ii) The right $\mathrm{End}_{R_{\mathscr{F}}} P_{\mathscr{F}}$-module $\mathrm{Hom}_{R_{\mathscr{F}}}(P_{\mathscr{F}}, M_{\mathscr{F}})$ can be made into a right $P_{\mathscr{F}}$ $\mathrm{End}_R P$-module by defining $xa = x\Phi_{P,P}(a)$, where $x \in \mathrm{Hom}_{R_{\mathscr{F}}}(P_{\mathscr{F}}, M_{\mathscr{F}})$, $a \in P_{\mathscr{F}} \mathrm{End}_R P$. Then $\Phi_{P,M}$ is a $P_{\mathscr{F}} \mathrm{End}_R P$-isomorphism.

Before the proof of the theorem we need the following lemmas.

Lemma 1.2. Let $P' \in \mathscr{F}(P)$ and $f \in \mathrm{Hom}_R(P', M)$. If $N \in \mathscr{F}(M)$, then $f^{-1}(N) \in \mathscr{F}(P)$.

Lemma 1.3. Let M be a $\mathscr{F}$-torsionfree R-module and $f \in \mathrm{Hom}_R(P, M)$. If ker $f \in \mathscr{F}(P)$ then $f=0$.

Proof. Let $x \in P$, then there exists an $I \in \mathscr{F}$ such that $xI \subseteq \ker f$; $f(x)I = f(xI) = 0$. Hence $f(x) \in t(M) = 0$.

Let $x \in P$, then there exists an R-homomorphism $x_L: R \to P/t(P)$, setting $a \in R$ to $xa + t(P)$. x_L representes an element in $P_{\mathscr{F}}$, which will be denoted by $\bar{x}$.

Lemma 1.4. Let $\{f_j\}_{j \in J}$ be an arbitrary set of representatives of $P_{\mathscr{F}} = \varinjlim \mathrm{Hom}_R(I, P/t(P))$. Then

$$\sum_{j \in J} \mathrm{Im} f_j \in \mathscr{F}(P/t(P)).$$

Proof. Assume that P is $\mathscr{F}$-torsionfree. Let $x \in P$, since $\{f_j\}_{j \in J}$ be a set of representatives of $P_{\mathscr{F}}$, the element $\bar{x}$ can be represented by $f_{j(x)}$, for some $j(x) \in J$; that is, x_L and $f_{j(x)}$ coincide on some $\mathscr{F}$-dense right ideal I_x contained in the domain of $f_{j(x)}$, $xI_x \subseteq \mathrm{Im} f_{j(x)}$. This shows that $\sum_{x \in P} xI_x \subseteq \sum_{j \in J} \mathrm{Im} f_j$, with $\sum_{x \in P} xI_x \in \mathscr{F}(P)$.

The proof of Theorem 1.1. Without loss of generality, we assume that M is $\mathscr{F}$-torsionfree. The mapping $\Phi_{P,M}$ can be defined as follows: Assume $x \in P_{\mathscr{F}}$ $\mathrm{Hom}_R(P, M)$ be represented by $\xi: P' \to M$, with $P' \in \mathscr{F}(P)$; then ξ induces $\bar{\xi}: P' + t(P)/t(P) \to M$. Define $\Phi_{P,M}(x) \in \mathrm{Hom}_{R_{\mathscr{F}}}(P_{\mathscr{F}}, M_{\mathscr{F}})$, which sets $a \in P_{\mathscr{F}}$, represented by

$f: I \to P/t(P)$; to the element of $M_{\mathscr{F}}$ represented by the composed homomorphisms

$$f^{-1}(P'+t(P)/t(P)) \xrightarrow{f} P'+t(P)/t(P) \xrightarrow{\xi} M;$$

it is easy to check that $\Phi_{P,M}$ is well defined, i.e. independent of the choices of representing homomorphisms ξ and f; and $\Phi_{P,M}$ is acturally an additive group homomorphism. If we make $\mathrm{Hom}_{R_{\mathscr{F}}}(P_{\mathscr{F}}, M_{\mathscr{F}})$ a right $P_{\mathscr{F}}\mathrm{End}_R P$-module as done in Theorem 1.1, then $\Phi_{P,M}$ is also an $P_{\mathscr{F}}\mathrm{End}_R P$-homomorphism. We claim that $\Phi_{P,M}$ has the properties stated in Theorem 1.1, To see this, it suffices to define a mapping

$$\Psi_{P,M}: \mathrm{Hom}_{R_{\mathscr{F}}}(P_{\mathscr{F}}, M_{\mathscr{F}}) \to P_{\mathscr{F}}\mathrm{Hom}_R(P, M)$$

such that $\Psi_{P,M} \cdot \Phi_{P,M} = \mathrm{Id}_{P_{\mathscr{F}}\mathrm{Hom}_R(P, M)}$, and $\Phi_{P,M} \cdot \Psi_{P,M} = \mathrm{Id}_{\mathrm{Hom}_{R_{\mathscr{F}}}(P_{\mathscr{F}}, M_{\mathscr{F}})}$.

Assume $x \in \mathrm{Hom}_{R_{\mathscr{F}}}(P_{\mathscr{F}}, M_{\mathscr{F}})$. For each $y \in P_{\mathscr{F}}$, let y and $x(y) \in M_{\mathscr{F}}$ be represented respectively by f_y and g_y. Without loss of generality we can assume that the domains of f_y and g_y coincide. Since $\{f_y\}_{y \in P_{\mathscr{F}}}$ is a set of representatives of $P_{\mathscr{F}}$; by Lemma 1.4 we have

$$\sum_{y \in P_{\mathscr{F}}} \mathrm{Im}\, f_y = P'/t(P) \in \mathscr{F}(P/t(P)).$$

Let x' be the R-homomorphisn

$$x': \quad \sum_{y \in P_{\mathscr{F}}} \mathrm{Im}\, f_y \to M,$$

$$\sum_{y \in P_{\mathscr{F}}} f_y(x_y) \to \sum_{y \in P_{\mathscr{F}}} g_y(x_y), \quad x_y = 0 \text{ for but a finite set}.$$

x' is well defined. Since, if $\sum_{y \in P_{\mathscr{F}}} f_y(x_y) = 0$ then $\sum_{y \in P_{\mathscr{F}}} y \cdot \overline{x}_y = 0$ in $P_{\mathscr{F}}$, with $\overline{x}_y \in R_{\mathscr{F}}$; hence $\sum_{y \in P_{\mathscr{F}}} x(y) \cdot \overline{x}_y = 0$; and $g = \sum_{y \in P} g_y(x_y)_L : R \to M$ is a representative of $\sum_{y \in P} x(y) \cdot \overline{x}_y$, thus there exists some $I \in \mathscr{F}$ such that $g|_I = 0$. By Lemma 1.3 we have $g = 0$. Hence $\sum_{y \in P_{\mathscr{F}}} g_y(x_y) = g(1) = 0$. Define $\Psi_{P,M}(x)$ in $P_{\mathscr{F}}\mathrm{Hom}_R(P, M)$ to be the element represented by the composed homomorphism

$$P' \xrightarrow{\pi} P'/t(P) = \sum_{y \in P_{\mathscr{F}}} \mathrm{Im}\, f_y \xrightarrow{x'} M.$$

where π is the canonical epimorphism. $\Psi_{P,M}(x)$ is independent of the choices of representing homomorphism of $P_{\mathscr{F}}$, It is routine to check that $\Psi_{P,M} \cdot \Phi_{P,M} = \mathrm{Id}_{P_{\mathscr{F}}\mathrm{Hom}_R(P, M)}$ and $\Phi_{P,M} \cdot \Psi_{P,M} = \mathrm{Id}_{\mathrm{Hom}_{R_{\mathscr{F}}}(P_{\mathscr{F}}, M_{\mathscr{F}})}$.

Proposition 1.5 Let R be a ring and let $\mathfrak{G}$ denote the Goldie topology on R R. If M_R is a nonzero nonsingular right R-module, then $P_{\mathfrak{G}}\mathrm{Hom}_R(M, R) \neq 0$.

Proof. Let $0 \neq m \in M$, then $r(m) \notin \mathfrak{G}$, thus there exists an nonzero right ideal I of R such that $r(m) \cap I = 0$. I is isomorphic to mI under the homomorpzism $a \to ma$, and the nonsingularity of M gives that $I \cap \overline{Z}(R) = \overline{Z}(I) = 0$, where $\overline{Z}$ is the Goldie torsion radical of Mod-R. By Zorn's Lemma there exists a submodule N of M such that $mI\cap N=0$ and $mI\oplus N\leqslant_e M$. Define $x\in P_{\mathfrak{G}}\mathrm{Hom}_R(M, R)$ to be repre sented by the composed homomorphism

$$mI+N\to mI\to I\to R/\overline{Z}(R),$$

then x is a nonzero element in $P_{\mathfrak{G}}\mathrm{Hom}_R(M, R)$.

Corollary 1.6. $R_{\mathfrak{G}}$ is a cogenerator of the Grothendieck category Mod-$(R,\mathfrak{G})$.

2. In this section the right maximal quotient ring of $\mathrm{End}_R P$, for a quasigenerator P_R, is discussed.

Definition 2.1. Let $\mathscr{F}$ be a Gabriel topology on R, and let P_R, M_R be right R-modules. We say that

P $\mathscr{F}$-generates M if and only if $\mathrm{Trace}\,_M P\in\mathscr{F}(M)$.

P is a $\mathscr{F}$-generator if and only if P $\mathscr{F}$-generates R;

P is a $\mathscr{F}$-quasigeneiator if and only if P $\mathscr{F}$-generates all the submodules of P^n, $n\in Z^+$.

If P $\mathscr{F}$-generates M and M $\mathscr{F}$-generates N, then P $\mathscr{F}$-generates N. Thus a $\mathscr{F}$-generator must $\mathscr{F}$-generates every R modules. If $\mathscr{F}\leqslant\mathscr{F}'$, then $P\mathscr{F}'$-generates M provided that P $\mathscr{F}$-generatos M. Henee a generator is an $\mathscr{F}$-generatr for every Gabriel topology $\mathscr{F}$ on R.

Example 2.2. Let $R=Z[x]$ and $P=(2,x)$, the ideal of R generated by 2 and x. Then P_R is a $\mathscr{D}$-generator but P_R is not a generator, where $\mathscr{D}$ is the dense topology on R.

Lemma 2.3. Let R be a ring and P_R be a right R-module. Let $\mathscr{F}^0_P$ denots the Gabriel topology on R cogenerated by the injective hull $E(P)$ of P. If P $\mathscr{F}^0_P$-generates each of it submodules, then

(i) For each $P'\in\mathscr{F}^0_P(P)$, $\mathrm{Hom}_R(P, P')=\{s\in\mathrm{End}_R P: s(P)\subseteq P'\}\subseteq\mathrm{End}_R P$ is a dense right ideal of $\mathrm{End}_R P$.

(ii) Conversely, if J is a dense right ideal of $\mathrm{End}_R P$ then

$$JP=\{\sum_i f_i(p_i): f_i\in J,\ p_i\in p\}\in\mathscr{F}^0_P(P).$$

Proof. (i) Let $P'\in\mathscr{F}^0_P(P)$, we will show that $\mathrm{Hom}_R(P, P')$ is dense in $\mathrm{End}_R P$. Let f, $0\neq g\in\mathrm{End}_R P$; then $P''=f^{-1}(P')\in\mathscr{F}^0_P(P)$ by Lemma 1.2. Since P $\mathscr{F}^0_P$ generates P'', $\mathrm{Trace}\,_{P''}P\in\mathscr{F}^0_P(P'')\subseteq\mathscr{F}^0_P(P)$, by Lemma 1.3, we have $g(\mathrm{Trace}\,_{P''}P)\neq 0$; hence there exists an $s\in\mathrm{Hom}_R(P, P'')\subseteq\mathrm{End}_R P$ such that $g(s(P))\neq 0$, and $fs(P)\subseteq f(P'')\subseteq P'$; i.e. $gs\neq 0$ and $fs\in\mathrm{Hom}_R(P, P')$.

(ii) Assume J is a dense right ideal of $\mathrm{End}_R P$. We will see that $JP\in\mathscr{F}^0_P(P)$. Suppose it is not the case, then there exists submodule P' of P such that $P'\supseteq JP$ and $\mathrm{Hom}_R(P'/JP, P)\neq 0$ (See Lemma vi .3.8, [1]). Let $0\neq f\in\mathrm{Hom}_R(P', P)$ be such that $f(JP)=0$. Since $\mathrm{Trace}\,_{P'}P\in\mathscr{F}^0_P(P')$, we have $f(\mathrm{Trace}_{P'}P)\neq 0$. Thus there exists a $g\in\mathrm{Hom}_R(P, P')\subseteq\mathrm{End}_R P$ such that $fg\neq 0$ ($fg\in$

$\mathrm{Eng}_R P$). If $s \in (J:g)$, then $gs \in J$ and $fgs(P) = f(gs(P)) \subseteq f(JP) = 0$, that is $fgs = 0$; $(fg)\cdot(J:g) = 0$, which contradicts the density of J.

Let P_R, M_R be right R-modules, and let $\mathscr{F}$ be a Gabriel topology on R. Then the following diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(P, M) & \xrightarrow{i} & P_{\mathscr{F}}\mathrm{Hom}_R(P, M) \\ & \searrow^{q} \qquad \nearrow^{\Phi_{P,M}} & \\ & \mathrm{Hom}_{R_{\mathscr{F}}}(P_{\mathscr{F}}, M_{\mathscr{F}}) & \end{array} \qquad (**)$$

commutes, where i is the canonical mapping, and q is the mapping $f \to f_{\mathscr{F}}$. If M is $\mathscr{F}$-torsionfree, then i is injective, and so is q. If we regard $P_{\mathscr{F}}\mathrm{Hom}_R(P, M)$, $\mathrm{Hom}_{R_{\mathscr{F}}}(P_{\mathscr{F}}, M_{\mathscr{F}})$ as canonical right $\mathrm{End}_R P$-modules, then all the mappings in the diagram above are $\mathrm{End}_R P$-homomorphisms.

Theorem 2.4. Let P_R be a right R-module such that P is a $\mathscr{F}_P^0$-quasigenerator. If M_R is a $\mathscr{F}_P^0$-torsionfree right R-module, then $\mathrm{Hom}_R(P, M) \to \mathrm{Hom}_{R_{\mathscr{F}_P^0}}(P_{\mathscr{F}_P^0}, M_{\mathscr{F}_P^0})$ is the localization of $\mathrm{Hom}_R(P, M)$ under the dense topology $\mathscr{D}$ of $\mathrm{End}_R P$. Specially, $\mathrm{End}_R P \to \mathrm{End}_{R_{\mathscr{F}_P^0}} P_{\mathscr{F}_P^0}$ is the maximal right quotient ring of $\mathrm{End}_R P$.

Proof. By Lemma 2.3 and Lemma 1.3, we know that $\mathrm{Hom}_R(P, M)$ is $\mathscr{D}$-torsionfree. Using Diagram (*), it suffices to show that $\mathrm{Hom}_R(P, M) \to P_{\mathscr{F}_P^0}\mathrm{Hom}_R(P, M)$ is the $\mathscr{D}$ localization of $\mathrm{Hom}_R(P, M)$. We will show this by two steps.

(i) There exists an $\mathrm{End}_R P$-monomorphism $\Psi: (\mathrm{Hom}_R(P, M))_{\mathscr{D}} \to P_{\mathscr{F}_P^0}\mathrm{Hom}_R(P, M)$ such that the following diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(P, M) & \xrightarrow{i} & P_{\mathscr{F}_P^0}\mathrm{Hom}_R(P, M) \\ & \searrow q \qquad \nearrow \Psi & \\ & (\mathrm{Hom}_R(P, M))_{\mathscr{D}} & \end{array}$$

commutes, where i, q are the canonical homomorphisms.

Let $x \in (\mathrm{Hom}_R(P, M))_{\mathscr{D}}$ be represented by $\xi: J_{\mathrm{End}P} \to \mathrm{Hom}_R(P, M)_{\mathrm{End}}$ with $J \in \mathscr{D}$. Then $JP \in \mathscr{F}_P^0(P)$ by Lemma 2.3. Let $\eta: JP \to M$, $\sum_{i=1}^{n} s_i(p_i) \to \sum_{i=1}^{n} \xi(s_i)(p_i)$; η is well defined; for, if $\sum_{i=1}^{n} s_i(p_i) = 0$ then for every $f \in \mathrm{Hom}_R(P, (p_1, \cdots, p_n)R)$ we have $\sum_{i=1}^{n} s_i \pi_i f = 0$, where π_i is the i-th projection $P^n \to P$; since ξ is an $\mathrm{End}_R P$-homomorphism and $\pi_i f \in \mathrm{End}_R P$ we have $\sum_{i=1}^{n} \xi(s_i)(\pi_i f) = 0$; but $\mathrm{Trace}_{(p_1, p_2, \cdots, p_n)R} P \in \mathscr{F}_P^0((p_1, p_2, \cdots, p_n)R)$, hence $\sum_{i=1}^{n} \xi(s_i)(p_i) = 0$ by Lemma 1.3. Define $\Psi(x) \in P_{\mathscr{F}_P^0}\mathrm{Hom}_R(P, M)$ to be represented by η, then Ψ is an $\mathrm{End}_R P$-monomorphism. Moreover, the diagram above commutes.

(ii) $i(\mathrm{Hom}_R(P, M))$ is a rational $\mathrm{End}_R P$-submodule of $P_{\mathscr{F}}\mathrm{Hom}_R(P, M)$. Let x, $0 \neq y \in P_{\mathscr{F}_P^0}\mathrm{Hom}_R(P, M)$ be represented by $\xi: P' \to M$ and $\eta: P'' \to M$ respectively. Then $\mathrm{Trace}_{P' \cap P''} P \in \mathscr{F}_P^0(P' \cap P'') \subseteq \mathscr{F}_P^0(P'')$, and $\eta(\mathrm{Trace}_{P' \cap P''} P) \neq 0$ by Lemma 1.3.

Hence there exists an $s\in \mathrm{Hom}_R(P, P'\cap P'')\in \mathrm{End}_R P$ such that $\eta s\neq 0$, and $\xi s\in \mathrm{Hom}_R(P, M)$. That is, $xs\in i(\mathrm{Hom}_R(P, M))$ and $ys\neq 0$.

With slight restriction of the terminology used in [7], we have the following corollary.

Corollary 2.5. (c.f. Theorem 3.5, [3]). Let P_R be a right R-module such that P generates each submodules of P^n, $n\in Z^+$. Let $S=\mathrm{End}_R P$, $H=\mathrm{End}_R(E(P))$. Then the following statements are equivalent.

(1) H is a right selfinjective ring and is isomorphic to S'_{max}.

(2) $H\cong S'_{max}(H\in\beta\to\beta|P_{\mathscr{F}_P^0})$.

(3) $H_s\cong E(S_s)$.

(4) $I(J)=0$ for every $J\in K(S)$ (Definition see [3]), where $I(J)$ denotes the left annihilator of J in S.

(5) $E(P_R)=E_{\mathscr{F}_P^0}(P_R)=P_{\mathscr{F}_P^0}$

(6) For any R-submodule M of P and any R-homomorphism $\alpha: M\to P$ there exists a rational submodule L of P and R-homomorphism $\beta: L\to P$ such that $M\subseteq L$ and $\beta|_M=\alpha$.

(7) $\mathrm{Ann}_S(M)=0$ for every $M\in K(P)$.

Example 2.6. Let D be a field and K be a proper subfield of D. Let $R=\begin{bmatrix} K & D\\ 0 & D\end{bmatrix}$, then R is a right nonsingular ring. Let $P=e_{11}R$, then P_R is a faithful nonsingular uniform right ideal of R with a minimal right ideal $N=\begin{bmatrix} 0 & D\\ 0 & 0\end{bmatrix}$, and $\mathrm{Hom}_R(P, N)=0$. $\mathrm{End}_R P\cong e_{11}R_{11}\cong K$. It is easy to see that $R'_{max}\cong M_2(D)$ and $P_{\mathscr{D}}\cong e_{11}R'_{max}$, $\mathrm{End}'_{max}(P_{\mathscr{D}})\cong D$; and the inclusion homomorphism $0\to K\to D$ is the canonical monomorphism $0\to \mathrm{End}_R P\to \mathrm{End}_{R_{max}}(P_{\mathscr{D}})$. This example shows that Theorem 2.4 is not true for right R-module which is not a quasigenrator, even in the case when P_R is finitely generated projective.

Remark. This example also gives a negative answer to the conjecture of Amistur (Remark 10.A, [4]).

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