Journal of Mathematical Research and Exposition
Vol.9 No.2, May, 1989

On the Accurate Order of Approximation of Some Classes of Functions*

Zhou Songping

(Dept.of Math., Hangzhou Univ.)

I. Introduction. Let $E_n(f)_{[a,b]}$ be the best approximation of degree n to $f(x)\in C_{[a,b]}$, $w_k(f,\delta)_{[a,b]}$ is the k-th modulus of smoothness of $f(x)$ in $[a,b]$.

In 1914, S.Bernstein [1] proved that $\lim_{n\to\infty} E_n(|x|)_{[-1,1]}$ exists. Since then, S. Bernstein, S.Nikol'skii, M. Hasson and the author worked in this direction to find the accurate order of approximation of some classes of functions. We don't mention the all results here, the details can be seen, for example, in the reference [2]. One of the recent results shows that (cf.[3]): Let $f(x)\in C^k_{[-1,1]}$, $w_2(f^{(k)},\delta)_{[-1,1]}=O(\varphi(\delta))$. If there exists $a\in(-1,1)$ such that

$$\lim_{h\to 0+}[\varphi(h)]^{-1}|f^{(k)}(a+h)+f^{(k)}(a-h)-2f^{(k)}(a)|>0,$$

then $E_n(f)_{[-1,1]}\sim n^{-k}\varphi(n^{-1})$,

where $\varphi(x)$ is some modulus function on $[0,\infty)$, which we shall define below.

Now we ask: If a is the endpoint of $[-1,1]$, is there a corresponding result for $E_n(f)_{[-1,1]}$? In some applications, this is an interesting subject. In this paper, we shall discuss the problem.

For convenience, in the following we are restricted to the interval $[0,1]$, and let $E_n(f)=E_n(f)_{[0,1]}$, $w_k(f,\delta)=w_k(f,\delta)_{[0,1]}$, $\|f\|=\|f\|_{C[0,1]}=\max_{0\le x\le 1}|f(x)|$,

$$\Delta_n(x)=\frac{\sqrt{x(1-x)}}{n}+\frac{1}{n^2},\quad \delta_h(x)=h\sqrt{x(1-x)}+h^2.$$

Φ is such a class of some functions $\varphi(x)$ satisfying the following properties:

(i) $\varphi(x)>0$, $x\in(0,\infty)$, $\varphi(x)\nearrow$ in some interval $[0,t_0]\subseteq[0,1]$, $\lim_{x\to 0+}\varphi(x)=0$,

(ii) there is a σ, $0<\sigma<2$ for some ε, x wilh $0<\varepsilon\le\varepsilon_0\le 1$ and $0<x\le x_0\le 1$ such that $\varphi(\varepsilon x)\ge\varepsilon^6\varphi(x)$,

(iii) $\delta^2\int_\delta^1\frac{\varphi(t)}{t^3}\,dt=O(\varphi(\delta))$.

Define

* Received May, 11, 1987.

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

$$F_k(f, a, \delta_h(a), \eta) = \int_{2^{-k}\delta_h(a)}^{2^{-k+1}\delta_h(a)} dx_1 \int_{x_1}^{2x_1} dx_2 \cdots \int_{x_{k-1}}^{2x_{k-1}} [f^{(k)}(a+\delta_h(a)+x_k)$$
$$+ f^{(k)}(a+\delta_h(a)-x_k) - f^{(k)}(a+\delta_h(a)+\eta x_k) - f^{(k)}(a+\delta_h(a)-\eta x_k)] dx_k .$$

2. We prove the following results.

Theorm I. Let $k \geqslant 0$, $\varphi(x) \in \Phi$, $f(x) \in C^k_{[0,1]}$. If $w_2(f^{(k)}, \delta) = O(\varphi(\delta))$ and there exist $a \in [0,1)$ and $\eta_0 > 0$ such that

(1) $\lim\limits_{h\to 0+} [\delta_h^k(a)\varphi(\delta_h(a))]^{-1} |F_k(f, a, \delta_h(a), \eta_0)| > 0,$

then there is a n-th algebraic polynomial $P_n(x)$ such that

(2) $$C_1 \leqslant \|[f(x) - P_n(x)] \Delta_n^{-k}(x) \varphi^{-1}(\Delta_n(x))\|_{c[a, a+M\triangle_n(a)]}$$
$$\leqslant \|[f(x) - P_n(x)] \Delta_n^{-k}(x)\varphi^{-1}(\Delta_n(x)\| \leqslant C_2 ,$$

where C_1, C_2 and M are positive constants independent of a and n.

Proof. From the pointwise estimate (cf.[4]) we can deduce that there is a n-th algebraic polynomial $P_n(x)$ such that

$$|f(x) - P_n(x)| \leqslant A\, \Delta_n^k(x) w_2(f^{(k)}, \Delta_n(x)),$$

so that the second part of inequality (2) is established. Now we establish the first part of (2). For $2^j \leqslant n \leqslant 2^{j+1}$, write

$$P_n(x) = P_n(x) - P_{2^j}(x) + \sum_{l=1}^{j} P_{2^l}(x) - P_{2^{l-1}}(x) + P_1(x),$$

then noticing

$$|P_{2^l}(x) - P_{2^{l-1}}(x)| \leqslant A[\Delta_{2^l}^k(x) w_2(f^{(k)}, \Delta_{2^l}(x)) + \Delta_{2^{l-1}}^k(x) w_2(f^{(k)},$$
$$\Delta_{2^{l-1}}(x)) \leqslant A_1 \Delta_{2^l}^k(x) w_2(f^{(k)}, \Delta_{2^l}(x)),$$

we have

$$|P_n^{(k+2)}(x)| \leqslant A_2[\Delta_n^{-2}(x)\varphi(\Delta_n(x)) + \sum_{l=1}^{j}[\Delta_{2^l}^{-2}(x)\varphi(\Delta_{2^l}(x))],$$

due to the properties (ii) and (iii) of $\varphi(x)$ it follows that

(3) $$|P_n^{(k+2)}(x)| \leqslant A_3 \Delta_n^{-2}(x)\varphi(\Delta_n(x)).$$

$$|F_k(P_n, a, \delta_h(a), \eta_0)| = |\int_{2^{-k}\delta_h(a)}^{2^{-k+1}\delta_h(a)} dx_1 \int_{x_1}^{2x_1} dx_2 \cdots \int_{x_{k-1}}^{2x_{k-1}} dx_k \int_{\eta_0 x_k}^{x_k} dx_{k+1} \int_{-x_{k+1}}^{x_{k+1}} P^{(k+2)}(a +$$
$$x_{k+2} + \delta_h(a)) dx_{k+2}| \leqslant 2\delta_h^{k+2}(a) \|P_n^{(k+2)}\|_{c[a, a+2\delta_h(a)]}$$

It is not difficult to see that for small h

$$\|\Delta_n^{-2}(x)\varphi(\Delta_n(x))\|_{c[a,a+2\delta_h(a)]} \leqslant A_4 \Delta_n^{-2}(a)\varphi(\Delta_n(a)),$$

hence by (3)

(4) $$|F_k(P_n, a, \delta_h(a), \eta_0)| \leqslant A_5 \delta_h^{k+2}(a)\, \Delta_n^{-2}(a) \varphi(\triangle_n(a)).$$

On the other hand, from the condition for small h we get

(5) $$|F_k(f, a, \delta_h(a), \eta_0)| \geqslant A_6 \delta_h^k(a) \varphi(\delta_h(a))$$
$$\geqslant A_6 \delta_h^k(a) \min\{1, [\delta_h(a)\Delta_n^{-1}(a)]^6\} \varphi(\Delta_n(a)).$$

Without lossing generality we assume that $A_5 \geqslant A_6$. Set $h = (A_6/(2A_5))^{1/(2-6)} h^{-1}$, combine (4) and (5)

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

$$|F_k(f, a, \delta_h(a), \eta_0) - F_k(P_n, a, \delta_h(a), \eta_0)|$$

$$\geqslant \frac{A_6}{2}\delta_h^k(a)\varphi(\Delta_h(a))[\delta_h(a)\Delta_n^{-1}(a)]^6 \geqslant A_7\Delta_h^k(a)\varphi(\Delta_n(a)).$$

Notice that when $h \sim n^{-1}$,

$$\|\Delta_n^k(x)\varphi(\Delta_n(x))\|_{c[a, a+2\delta_h(a)]} \leqslant A_8\Delta_n^k(a)\varphi(\Delta_n(a)),$$

therefore there exists a $x^* \in [a, a+2\delta_h(a)]$ such that

$$\|f - P_n\|_{c[a, a+2\delta_h(a)]} = |f(x^*) - P_n(x^*)|,$$

so

$$A_9\Delta_n^k(x^*)\varphi(\Delta_n(x^*)) \leqslant A_7\Delta_n^k(a)\varphi(\Delta_n(a))$$
$$\leqslant |F_k(f, a, \delta_h(a), \eta_0) - F_k(P_n), a, \delta_h(a), \eta_0)| \leqslant A_{10}|f(x^*) - P_n(x^*)|,$$

thus the proof is completed.

For convenient application, we raise the following

Theorem 2. Let $k \geqslant 0, \varphi(x) \in \Phi$, $f(x) \in C^k_{[0,1]}$. If $w_2(f^{(k)}, \delta) = O(\varphi(\delta))$ and there exists $a \in [0,1)$ for $2^{-k}\delta_h(a) \leqslant t \leqslant \delta_h(a)$ such that

$$(6)\quad \lim_{h\to 0+}[\varphi(t)]^{-1}|f^{(k)}(a+\delta_h(a)+t) + f^{(k)}(a+\delta_h(a)-t) - 2f^{(k)}(a+\delta_h(a))| > 0.$$

then there is a n-th algebraic polynomial $P_n(x)$ such that

$$C_1 \leqslant \|[f(x) - P_n(x)]\Delta_n^{-k}(x)\varphi^{-1}(\Delta_n(x))\|_{c[a, a+M\triangle_n(a)]}$$
$$\leqslant \|[f(x) - P_n(x)]\Delta_n^{-k}(x)\varphi^{-1}(\Delta_n(x))\| \leqslant C_2.$$

It is a corollary of Theorem 1, the details of proof is omitted here.

Let $P_n(f, x)$ be the n-th polynomial of best approximation to $f(x) \in C_{[0,1]}$. From the proof of Theorem 1. we can assert if $f(x) - P_n(f, x) = O(\triangle_n^k(x)\varphi(\triangle_n(x)))$ and the condition (1) is satisfied, then the inequality (2) is valid. Therefore we give the following theorem.

Theorem 3. Let $k \geqslant 0, \varphi(x) \in \Phi$, $f(x) \in C^k_{[0,1]}$. Define $g(x) = f(x^2)$. If $g(x) \in C^{2k}_{[-1,1]}$, $w_2(g^{(2k)}, \delta) = O(\varphi(\delta^2))$, then

$$E_n(f) \sim n^{-2k}\varphi(n^{-2})$$

It's obvious that

$$E_n(g)_{[-1,1]} = E_n(f).$$

so

$$|f(x) - P_n(f, x)| \leqslant E_n(f) \leqslant Cn^{-2k}w_2(g^{(2k)}, n^{-1}) \leqslant C\triangle_n^k(x)w_2(g^{(2k)}, \triangle_n^{1/2}(x)),$$

the other part of the proof can be deduced from Theorem 1 or Theorem 2.

3. Application. Let $\beta = r + a$.

$$f_\beta(x) = \begin{cases} |x|^\beta, & \text{if } r \text{ is even}, \\ x|x|^{\beta-1}, & \text{if } r \text{ is odd}, \end{cases} \quad a \in (0, 1/2) \cup (1/2, 1),$$

applying Theorem 3, we get

$$E_n(f_\beta) \sim n^{-2\beta}.$$

(to 305)

Again

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

References

[1] Attele, K. R. M., Analytic multipliers of Bergman spaces, Michigan Math. J. 31(1984), 307—319.

[2] Carleson, L., Interpolation by bounded analytic functions and the corona problem, Ann. of Math. 76(1962), 547—559.

[3] Duren, P.L., Theory of H^p Spaces, Academic Press, New York, 1970.

[4] Garnett, J.B., Bounded Analytic Functions, Academic Press, New York, 1981.

[5] Hastings W.W. A Carleson measure for Bergman spaces, Proc. Amer. Math. Soc. 52(1975), 237—241.

[6] Taylor, G. D., Multipliers on D_a, Trans. Amer. Math. Soc. 123(1966), 229—240.

[7] Xiao Jianbin, Some properties of conjugate functions, to appear.

[8] Zygmund, A., Trigonometric series, 2nd ed., Cambridge University Press, Cambridge, 1959.

(from 308)

$$g_{\beta,\gamma}(x) = \begin{cases} |x|^{\beta}|\ln^{\gamma} x|, & \text{if } r \text{ is even}, \\ x|x|^{\beta-1}|\ln^{\gamma} x|, & \text{if } r \text{ is odd}, \end{cases} \quad 0<a<1, \gamma>0,$$

then

$$E_n(g_{\beta,\gamma}) \sim n^{-2\beta}\ln^{\gamma} n .$$

References

[1] S.Bernstein, Sur la meilletre approximation de $|x|$ par des polynemes de degrés donnés, Acta Math., 37 (1914), 1-57.

[2] M. Hasson, Functions f for which $E_n(f)$ is exactly of the order n^{-1} in "Approximation Theory III", AP (1980), New York, 491—494.

[3] С. П. Чжоу, О доказательстве одной теоремы Хассона, Вестн. Белорус. ун—та. to appear.

[4] В. К. Дзядык, Введение в теорию равномерного приближения функций, М. . 1977 г., 512 стр..

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.