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A Fixed Point Theorem and its Application to Best Approximation*

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The object of this paper is twofold. First, a fixed point theorem of G.L. Cain, Jr. and M.Z.Nashed [1] is generalized. Second, the theorem (Theoreml) is utilized to obtain theorems on best approximation which extend and unify the results of Meinardus [2], Singh [3—4], and Sahney-Singh-Whitfield [5].

Throught this paper E will denote a Hausdorff locally convex linear topological space "T_2LCS" and Q a (fixed) family of continuous seminorms which generates the topology of E. Let G be a nonempty subset of E, and p be a continuous seminorn on E. For $x\in$E, define

$$d_p(x, G)=\inf\{p(x-y):y\in G\}.$$

A mapping T:G→G is said to be a p-contractive if there is a λ_p, $0\leqslant\lambda_p<1$ such that

$$p(Tx-Ty)\leqslant\lambda_p p(x-y) \qquad (1)$$

for all $x, y\in$G and $p\in$Q.

Definition 1. A mapping T on a subset G of a T_2LCS E, mapping G into E is said to be p-locally contractive if for every $x\in$G and $p\in$Q there exist $\varepsilon_p(x)$ and $\lambda_p(x)$ ($\varepsilon_p>0, 0\leqslant\lambda_p<1$) such that:

$$y_1, y_2\in S_p(x, \varepsilon_p)=\{g: p(g-x)\leqslant\varepsilon_p\} \text{ implies (1)} \qquad (2)$$

If both ε_p and λ_p do not depend on $x\in$G, T is said to be $(\varepsilon_p, \lambda_p)$-uniformly locally contractive.

Definition 2. A subset G of E is said to be ε_p-chainable if for any pair x, y $\in$G and all $p\in$Q, we can find a finite number of elements x_i, $i=0,1,2,\cdots,k$ with $x_0=x$ and $x_k=y$ such that:

$$p(x_{i-1}-x_i)<\varepsilon_p, \quad i=1,2,\cdots,k.$$

Theorem 1. Let G be a sequentially complete ε_p-chainable subset of a T_2LCS E. If T:G→G is $(\varepsilon_p, \lambda_p)$-uniformly locally contractive, then there exists a unique point $u\in$G such that $Tu=u$.

Theorem 2. Let E be a T_2LCS, G be a subset of E, and T: E→E having

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a fixad point b such that $p(x-y)\leqslant\varepsilon_p$(for a positive number ε_p, $\varepsilon_p\geqslant d_p(b,G)$ implies $p(Tx-Ty)\leqslant p(x-y)$ for every $p\in Q$. Let T map ∂G into G. Assume that for every $p\in Q$ the set D of best G approximants to b with respect to p is none mpty, sequentially complete, bounded, and starshaped. Furthermore, assume that $(I-T)(D)$ is closed. Then T has a fixed point which is a best approximation to b in G.

Results of Meinardus [2,Theorem], Singh [4,Theorem 1], and Sahney Singh Whitfield [5,Theorem 2.1] can be easily deduced from Theorem 2.

In the following, we try to relax the starshaped condition on D in Theorem 2.

Definition 3. Let D be a closed subset of E. A family of maps $\{f_x\}_{x\in D}$ is said to be quasiconvex structure on D if it satisfies the following conditions:

(i) $f_x:[0,1]\to D$, i.e., f_x is a map from [0,1] into D for each $x\in D$.

(ii) $f_x(1)=x$ for each $x\in D$.

(iii) $f_x(t)$ is jointly continuous in (x,t) i.e. $f_x(t)\to f_{x_0}(t_0)$ for $x\to x_0$ in D and $t\to t_0$ in [0,1].

(iv) $p(f_x(t)-f_y(t))\leqslant\Phi_p(t)\,p(x-y)$ for all $x,y\in D$ and $p\in Q$, where Φ_p is a function from [0,1] into itself.

Theorem 3. Let E be a T_2LCS, and T:E→E be p nonexpansive mapping. Let $T:\partial G\to G$ and b be a T invariant point. Assume that for every $p\in Q$ the set D of best G approximants to b with respect to p is nonempty and compact. Furthermore, assume that there exists a quasiconvex structure on D. Then T has a fixed point which is a best approximation to b in G.

References

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