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On n-Dimensional Matrix Groups*

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Abstract An $n\times$matrix $A=\{a_{ij}\}$ will be called a 2-dimensional matrix of order n. We shall study groups of $2m$-dimensional matrices of order n over a field of characteristic O with respect to an associative matrix product and we obtain the dimensions of such matrix groups.

Introduction There are several papers on n-dimensional matirces (cf [3,4,5,6,7]). Sokolov [7] is significant. Oldenburger [6] considered determinants of n-dimensional matrices. We describe some of the results of this paper. Let R, C, and H denote respectively the real number field, the complex number filed and the field of the real quaternions. Let $M_{2m,n}(F)$ denote the set of all $2m$-dimensional matrices of order n over a field $F\in\{R, C, H\}$. In $M_{2m,n}(F)$ there is an associative matrix product (see [5]) and under this matrix product $M_{2m,n}(F)$ forms a matrix group (and ring). We consider $M_{m,n}(F)$ as a vector space and we define an inner product $\langle X, Y\rangle$ for X, Y in $M_{m,n}(F)$. Then we can have a matrix group $S(2m, n, F)=\{A\in M_{2m,n}(F):\langle XA, YA\rangle$ for all $X, Y\in M_{m,n}(F)\}$. If G is a group, its dimension is the dimension of the vector space $T(G)$ of the tangent vectors to G at the identity I. To get the dimension of the group $S(2m, n, F)=S(F)$ we need $T(S(F))$. We shall prove that $T(S(F))=\mathrm{So}(2m, n, F\}=\{A, \in M_{2m,n}(F): A+A^{-t}=0\}$. where A^{-t} denotes the transpose and conjugate matrix of A. We find the dimension of $T(S(F))$, the vector space over the real numbers R, using a method of *'one parameter subgroups'*.

I. A Matrix product and an Inner Product of Two Vectors

We begin with the following definitions.

Definition I[5] Let $A=(a_{i_1i_2\cdots i_{2m}})$ and $B=(b_{i_1i_2\cdots i_{2m}})$ be two matrices in $M_{2m,n}(F)$. We define a matrix product $AB=C=(c_{i_1i_2\cdots i_{2m}})$ as follows:

$$c_{i_1i_2\cdots i_{2m}}=\sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\cdots\sum_{t_m=1}^{n}a_{i_1i_2\cdots i_m t_1t_2\cdots t_m}b_{t_1t_2\cdots t_m i_{m+1}\cdots i_{2m}}$$

Note that this matrix product is associative [5] and under this matrix product,

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$M_{2m,n}(F)$ forms a semigroup (and a ring).

Definition 2 The inner product of two vectors $X=(x_{ij\cdots k})$ and $Y=(y_{ij\cdots k})$ in $M_{m,n}(F)$ is defined as follows:

$$\langle X, Y\rangle = \sum_{t_i=1}^{n} x_{t_1 t_2\cdots t_m}\overline{y}_{t_1 t_2\cdots t_m} \quad (i=1,2,\cdots,n)$$

where $\overline{y}$ denotes the conjugate of $y\in C$ or $y\in H$.

We list the following two lemmas without proofs.

Lemma 1 The inner product $\langle X, Y\rangle$ of two vectors X and Y in $M_{m,n}(F)$ has the following properties.

(i) $\langle X, Y+Z\rangle = \langle X, Y\rangle + \langle X, Z\rangle$, $\langle X+Y, Z\rangle = \langle X, Z\rangle + \langle Y, Z\rangle$ for all X, Y, and $Z\in M_{m,n}(F)$.

(ii) $a\langle X, Y\rangle = \langle aX, Y\rangle$ and $\langle X, Y\rangle\overline{a} = \langle X, aY\rangle$ for all $a\in F$ and $X, Y\in M_{m,n}(F)$.

(iii) $\overline{\langle X, Y\rangle} = \langle Y, X\rangle$ for all $X, Y\in M_{m,n}(F)$.

(iv) $\langle X, Y\rangle \geqslant 0$ and $\langle X, X\rangle = 0$ iff $X=0$.

Lemma 2 For any $X, Y\in M_{m,n}(F)$ and $A\in M_{2m,n}(F)$, we have that $\langle XA, Y\rangle = \langle X, YA^{-t}\rangle$.

From now on F denotes a field in $\{R, C, H\}$. We prove the following theorem.

Theorem 3 Let $S(2m, n, F) = \{A\in M_{2m,n}(F): \langle XA, YA\rangle = \langle X, Y\rangle$ for all $X, Y\in M_{m,n}(F)\}$. Then $S(2m, n, F)$ is a group.

Proof Let $A=(a_{ij\cdots k})$, $B=(b_{ij\cdots k})\in S(2m, n, F)$. Then we see that $\langle XAB, YAB\rangle = \langle XA, YA\rangle = \langle X, Y\rangle$ for all $X, Y\in M_{m,n}(F)$, and hence $AB\in S(2m, n, F)$. We know that there exists $I=(x_{i_1 i_2\cdots i_{2m}})$, which is defined by (see [4]).

$$x_{i_1 i_2\cdots i_{2m}} = \begin{cases} 1 & \text{if } (i_1 i_2\cdots i_m) = (i_{m+1} i_{m+2}\cdots i_{2m}), \\ 0 & \text{otherwise,} \end{cases}$$

such that $XI = IX = X$ and $AI = IA = A$ for all $X\in M_{m,n}(F)$ and $A\in M_{2m,n}(F)$. Thus $I\in S(2m, n, F)$. We use $I=(\delta_{i_1 i_2\cdots i_m}^{i_{m+1} i_{m+2}\cdots i_{2m}})$ for the identity I. where δ denotes the Kronecker's delta. Let $e_{i_1 i_2\cdots i_m}$ be a member of $M_{m,n}(F)$ whose $(i_1 i_2\cdots i_m)$-entry is 1 and others are zero. We consider $\langle e_{i_1 i_2\cdots i_m}A, e_{j_1 j_2\cdots j_m}A\rangle$.

We first see that

$$\langle e_{i_1 i_2\cdots i_m}A, e_{i_1 j_2\cdots j_m}A\rangle = \delta_{i_1 i_2\cdots i_m}^{j_1 j_2\cdots j_m},$$

and also see that

$$\begin{aligned}\langle e_{i_1 i_2\cdots i_m}A, e_{j_1 j_2\cdots j_m}A\rangle &= \langle (a_{i_1 i_2\cdots i_m 1 1\cdots 1}, \cdots, a_{i_1 i_2\cdots i_m k_1 k_2\cdots k_m}, \cdots, a_{i_1 i_2\cdots i_m n n\cdots n}), \\ &\quad (a_{j_1 j_2\cdots j_m 1 1\cdots 1}, a_{j_1 j_2\cdots j_m k_1 k_2\cdots k_m}, \cdots, a_{j_1 j_2\cdots j_m n n\cdots n})\rangle \\ &= \sum_{k_t=1}^{n} a_{i_1 i_2\cdots i_m k_1 k_2\cdots k_m}\overline{a}_{k_1 k_2\cdots k_m j_1 j_2\cdots j_m}, \quad (t=1,2,\cdots,n).\end{aligned}$$

which is the $(i_1 i_2\cdots i_m j_1 j_2\cdots j_m)$-entry of AA^{-t}. Hence this entry must be 1 if $i_p=$

j_p $(p=1,2,\cdots,m)$ or 0 if $i_p \neq j_p$.

Therefore $AA'=I$. We can also see that $A'A=I$. Thus $A^{-\prime}=A^{-1}$. Now we see that $\langle XA^{-1}, XA^{-1}\rangle=\langle XA^{-1}A, YA^{-1}A\rangle=\langle X, Y\rangle$. This proves Theorem 3. We may call $S(2m, n, F)$ an orthogonal group.

Note Referring the identity I in the proof of Theorem 3, we define $GL(2m, n, F)=\{A\in M_{2m,n}(F): AA^{-1}=I\}$ as the general linear group in $M_{2m,n}(F)$.

2. Dimensions of Vector Spaces $\mathrm{So}(2m, n, F)$.

In this section we compute dimensions of vector spaces $\mathrm{So}(2m, n, F)$ over the real numbers R.

Definition 3 [1] We define $\mathrm{So}(2m, n, F)=\{A\in M_{2m,n}(F): A+A^{-\prime}=0\}$. A matrix A in $\mathrm{So}(2m, n, F)$ is called a F-skew symmetric matrix.

Lemma 4 $\mathrm{So}(2m, n, F)$ is a vector space over R.

Proof We see that the zero matrix 0 is in $\mathrm{So}(2m, n, F)$. Let $A, B\in\mathrm{So}(2m, n$ $F)$. Then we see that $(A+B)+(\overline{A+B})'=0$ and hence $A+B\in\mathrm{So}(2m, n, F)$. We also see that $rA+(\overline{rA})'=r(A+A^{-\prime})=0$ for all $r\in R$. Therefore $rA\in\mathrm{So}(2m, n, F)$. This proves the lemma.

Dim V denotes the dimension of a vector space V (over R).

Theorem 5 (i) Dim $\mathrm{So}(2m, n, R)=n^m(n^m-1)/2$.

(ii) Dim $\mathrm{So}(2m, n, C)=n^{2m}$.

(iii) Dim $\mathrm{So}(2m, n, H)=n^m(2n^m+1)$.

Proof We prove (i). To get the dimension of $\mathrm{So}(2m, n, R)$ we must find a basis. Let $E_{i_1i_2\cdots i_mj_1j_2\cdots j_m}$ denote the matrix in $M_{2m,n}(R)$ whose entries are all zero except the $(i_1i_2\cdots i_mj_1j_2\cdots j_m)$ entry, which is 1, and the $(j_1j_2\cdots j_mi_1i_2\cdots i_m)$ entry, which is $-1((i_1i_2\cdots i_m)\neq(j_1j_2\cdots j_m))$. Then we see that $E_{i_1i_2\cdots i_mj_1j_2\cdots j_m}\in\mathrm{So}(2m, n, R)$. We redefine these $E_{i_1\cdots i_mj_1\cdots j_m}$ only for $i_t<j_t$, it is easy to see that they form a basis for $\mathrm{So}(2m, n, R)$, and it is not difficult to count that there are $(n^m-1)+(n^m-2)+\cdots+1=n^m(n^m-1)/2$ of them. This proves that dim $\mathrm{So}(2m, n, R)=n^m(n^m-1)/2$.

For (ii), Let $B=(b_{i_1\cdots i_mj_1\cdots j_m})$ be a matrix in $\mathrm{So}(2m, n, C)$. Then $B+B^{-\prime}=0$, from which we get that $b_{i_1i_2\cdots i_mj_1j_2\cdots j_m}+\overline{b}_{j_1j_2\cdots j_mi_1i_2\cdots i_m}=0$ for any $(i_1i_2\cdots i_mj_1j_2\cdots j_m)$ entry of B. If $b_{i_1i_2\cdots i_mj_1j_2\cdots j_m}=c+di$, then $\overline{b}_{j_1j_2\cdots j_mi_1i_2\cdots i_m}=-c-di$ and $b_{j_1j_2\cdots j_mi_1i_2\cdots i_m}=-c+di$. If $(i_1i_2\cdots i_m)=(j_1j_2\cdots j_m)$, then $c+di=-c+di$ and $c=0$. Consequently, the number of elements consisting a basis is equal to $n^m+2n^m(n^m-1)/2=n^{2m}$.

For (iii), letting $C=(c_{i_1i_2\cdots i_mj_1\cdots j_m})\in S(2m, n, H)$, we have that $C+C^{-\prime}=0$ and $c_{i_1\cdots i_mj_1\cdots j_m}+\overline{c}_{j_1j_2\cdots j_mi_1i_2\cdots i_m}=0$ for any $(i_1i_2\cdots i_mj_1j_2\cdots j_m)$ entry of C. If $c_{i_1i_2\cdots i_mj_1j_2\cdots j_m}=x+yi+uj+vk$, then $\overline{c}_{j_1\cdots j_mi_1\cdots i_m}=-x-yi-uj-vk$ and $c_{j_1\cdots j_mi_1\cdots i_m}=-x+yi+uj+vk$. If $(i_1i_2\cdots i_m)\neq(j_1j_2\cdots j_m)$ then $x=0$ and $c_{i_1\cdots i_mj_1\cdots j_m}=yi+uj+vk$. Consequently, the number of elements consisting a basis is equal to $3n^m+4(n^m(n^m-1)/2=n^m(2n^m+1)$.

This proves Theorem 5.

3. Curves in a Vector Space

We need the following definition.

Definition 4 Let V be a finite dimensional vector space. π is called a curve in V if $\pi:(a,b)\to V$ is a continuous function, where (a,b) is an open interval in R. For $c\in(a,b)$, we say that π is differentiable at c if

$$\lim_{h\to 0}\frac{\pi(c+h)-\pi(c)}{h}$$

exists. When this limit exists, it is a vector in V. We denote it by $\pi'(c)$ and call the tangent vector to π at $\pi(c)$. Note that $M_{2m,n}(R)$, $M_{2m,n}(C)$ and $M_{2m,n}(H)$ are real vector spaces of dimensions n^{2m}, $2n^{2m}$ and $4n^{2m}$, respectively. If G is a matrix group in $M_{2m,n}(F)$, then a curve in G is a curve in $M_{2m,n}(F)$ with all values $\pi(t)$ for $t\in(a,b)$ lying in G. If π_1 and π_2 are curves such that $\pi_i:(a,b)\to G$ then the product curve is defined by $(\pi_1\pi_2)(t)=\pi_1(t)\pi_2(t)$, for $t\in(a,b)$.

Theorem 6 Let G be a matrix group in $M_{2m,n}(F)$. Let T be the set of all tangent vectors $\pi'(0)$ to the curves $\pi:(-s,s)\to G$ with $\pi(0)=I$, the identity matrix of G. Then T is a subspace of $M_{2m,n}(F)$.

Proof Let $\pi_i'(0)\in T$ $(i=1,2)$. Then $(\pi_1\pi_2)(0)=I$, $(\pi_1\pi_2)'(0)=\pi_1'(0)\pi_2(0)+\pi_1(0)\pi_2'(0)=\pi_1'(0)+\pi_2'(0)\in T$, and hence T is closed under vector addition. Let $\pi_1'(0)\in T$ and $c\in$R. Define $\pi_2(t)=\pi_1(ct)$. Then we see that $\pi_2(0)=\pi_1(0)=I$ and $\pi_2'(0)=c\pi_1'(0)$ since $\pi_2'(t)=c\pi_1'(t)$. This proves Theorem 6.

Definition 5 ([1, p.37]) If G is a matrix group, its dimension is the dimension of the vector space $T(G)$ which is the set of all tangent vectors of G at I.

Let T be the set of all tangent vectors defined as in Theorem 6.

Theorem 7 If π is a curve through the identity, that is, $\pi(0)=I$, then $\pi'(0)\in T$ is a F-skew symmetric matrix in So$(2m,n,F)$.

Proof By differentiating both sides of $\pi(u)\pi^{-1}(u)=I$, we obtain that $\pi'(u)\pi^{-1}(u)+\pi(u)(\pi^{-1})'(u)=0$ and $\pi'(0)+(\pi^{-1})'(0)=0$, which shows that $\pi'(0)$ is F-skew symmetric. This proves Theorem 7.

Corollary

(i) dim $S(2m,n,R)\leq n^m(n^m-1)/2$.

(ii) dim $S(2m,n,C)\leq n^{2m}$.

(iii) dim $S(2m,n,H)\leq n^m(2n^m+1)$.

Proof By Theorem 3, $S(2m,n,F)$ is a matrix group. We see that dim $S(2m,n,F)=$ dim $T(S(2m,n,F))=$ dim$\{\pi'(0):\pi(u)\in S(2m,n,F)\}$. Since $\pi'(0)\in$ So$(2m,n,F)$, dim $S(2m,n,F)$ can not exceed dim so$(2m,n,F)$, $F\in\{R,C,H\}$. This proves Corollary.

Let $\varphi:G_1\to G_2$ be a homomorphism of matrix groups G_1 and G_2. Since G_i are in vector spaces, it is clear what it means for φ to be continuous. From now on homomorphism always means continuous homomorphism. This being so, a curve $\pi:(a,b)\to G_1$ gives a curve $\varphi\pi:(a,b)\to G_2$ by $(\varphi\pi)(u)=\varphi(\pi(u))$ in G_2.

Definition 6 A homomorphism $\varphi:G_1\to G_2$ of matrix groups is smooth if for every differentiable curve π in G_1, $\varphi\pi$ is differentiable. This definition is needed in the next section.

4. A One Parameter Subgroup in a Matrix Group

Using Definition 6, we define '*one parameter subgroup*' as follows.

Definition 7 A one parameter subgroup π in a matrix group $G(\subset M_{2m,n}(F))$ is a smooth homomorphism $\pi:R\to G$ with $\pi(s+t)=\pi(s)\pi(t)$ for all s and t in R.

Note that it suffices to know π on some open neighborhood U of 0 in R. For $x\in R$, some $\frac{1}{n}x\in U$ and $\pi(x)=(\pi(\frac{1}{n}x))^n$.

Example Let $A\in M_{2m,n}(F)$. Then $\pi(u)=e^{uA}$ is a one parameter subgroup of $GL(2m,n,F)$. (We often use t instead of u).

Theorem 8 π is a one parameter subgroup of $GL(2m,n,F)$ iff there exists A in $M_{2m,n}(F)$ such that $\pi(t)=e^{tA}$ with $\pi'(0)=A$.

Proof Suppose π is a one parameter subgroup of $GL(2m,n,F)$, and let $\log\pi(t)=\lambda(t)$. Then λ is a curve in $M_{2m,n}(F)$ with $\pi(t)=e^{\lambda(t)}$. Set $\lambda'(0)=A$. Then for any fixed t_0,

$$\lambda'(t_0)=\lim_{h\to 0}\frac{\lambda(t_0+h)-\lambda(t_0)}{h}=\lim_{h\to 0}\frac{\log\pi(t_0+h)-\log\pi(t_0)}{h}$$

$$=\lim_{h\to 0}\frac{\log\pi(t_0)\pi(h)-\log\pi(t_0)}{h}=\lim_{h\to 0}\frac{\log\pi(h)}{h}=\lambda'(0)=A.$$

This means that $\lambda'(t)$ is independent of t. Hence, $\lambda(t)=tA$ which implies that $\lambda(t)$ is a line thgrough $0\in M_{2m,n}(F)$ and $\pi(t)=e^{tA}$. Conversely, suppose that there exists λ in $M_{2m,n}(F)$ such that $\pi(t)=e^{tA}$ with $\pi'(0)=A$. Then

$$\pi(t)=I+A+\frac{t^2A^2}{2!}+\frac{t^3A^3}{3!}+\cdots,\ \pi(0)=I,\ \pi(-t)=I-tA+\frac{t^2A^2}{2!}-\cdots$$

and $\pi(t)\pi(-t)=I$. This means that $\pi(t)\in GL(2m,n,F)$. Finally, we see that $\pi(t+s)=\pi(t)\pi(s)$. This proves Theorem 8. This theorem shows that any tangen vector to $GL(2m,n,F)$ is the derivative at 0 of some one parameter subgroup.

Lemma 9 If $A\in So(2m,n,F)$, then $e^{uA}\,e^{(\overline{uA})'}=I$, where $u\in R$.

Proof By the assumption, $A+A^{-\prime}=0$ for $A\in So(2m,n,F)$. Then we see that

$$e^{uA}e^{(\overline{uA})'}=I+u(A+A^{-\prime})+\cdots+u^n\Big(\sum_{k=0}^{n}\frac{A^{n-k}(A^{-\prime})^k}{(n-k)!\,k!}\Big)+\cdots=\sum_{n=0}^{\infty}I_nu^n,$$

where $$I_0=I,\ I_n=\sum_{k=0}^{n}\frac{A^{n-k}(A^{-\prime})^k}{(n-k)!\,k!}.$$

We prove that $I_n=0$ for $n>1$ because $I_1=0$. We prove it by mathematical induction on n. We assume that we have proven that $I_t=0$ for all $t<n$, where n is a fixed integer greater than 1. We define

$$\binom{k}{n}=\frac{(n-1)(n-2)\cdots(n-k)}{n!\,k!} \quad \text{and} \quad I(k)=\frac{A^{n-k}(A^{-t})^k}{(n-k)!\,k!}.$$

Then we see that

$$I(n)=\sum_{k=0}^{n} I(k)\sum_{k=0}^{m} I(k)+I(m+1)+I(m+2)+\cdots+I(n)$$

$$=\binom{m}{n}A^{n-m}(A^{-t})^m+I(m+1)+I(m+2)+\cdots+I(n)=\sum_{k=0}^{n-1}I(k)+I(n)$$

$$=\frac{(n-1)!}{n!\,(n-1)!}A(A^{-t})^{n-1}+\frac{(A^{-t})^n}{n!}=\frac{1}{n!}(A+(A^{-t}))(A^{-t})^{n-1}=0,$$

that is, $I(n)=0$. Consequently, $e^{uA}e^{\overline{(uA)}'}=I=e^0$. This proves Lemma 9.

Theorem 10 Let A be a tangent vector to the orthogonal group $S(2m,n,F)$. Then there exists a unique one parameter subgroup π in $S(2m,n,F)$ such that $A=\pi'(0)$.

Proof We have that $A=\pi'(0)$ for some curve π in $S(2m,n,F)$ since A is a tangent vector to the orthogonal group $S(2m,n,F)$. For any $u\in(-t_0,t_0)$ we have $\pi(u)\overline{\pi(u)}'=I$ and $\pi'(u)\overline{\pi(u)}'+\pi(u)(\overline{\pi}(u'))'=0$. Letting $u=0$, we obtain that $\pi'(0)+(\pi^{-t})'(0)=0$ which is equivalent to $A+A^{-t}=0$. Thus $A\in \mathrm{So}(2m,n,F)$. Now $\pi(u)=e^{uA}$ is a one-parameter subgroup of $\mathrm{GL}(2m,n,F)$ by Theorem 8, but it lies in $S(2m,n,F)$ because $\pi(u)(\overline{\pi(u)})'=e^{uA}e^{u(A')}=I$ by Lemma 9. This proves Theorem 10.

Now we obtain the following proposition.

Proposition 11 ([1, p.53—p.54]).

(i) $\dim S(2m,n,R)=\dim \mathrm{So}(2m,n,R)=\dfrac{n^m(n^m-1)}{2}$.

(ii) $\dim S(2m,n,C)=\dim \mathrm{So}(2m,n,C)=n^{2m}$.

(iii) $\dim S(2m,n,H)=\dim \mathrm{So}(2m,n,H)=n^m(2n^m+1)$.

Proof It follows from Definition 7, Theorem 5 and Theorem 10.

5. A Problem

To pose a problem we need a definition.

Definition 8 ([1, p.93 and p. 127]) A k-torus is the cartesian product of k circle groups. The rank of a matrix group G is the dimension of a maximal torus in G.

Problem Let $F\in\{R,C,H\}$. Find ranks of $S(2m,n,F)$.

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