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A Note on Ideals of a Three-Dimensional Associative Algebra*

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In this paper, we shall apply the theory of nilpotent radical to derive some results about ideals of a there-dimensional associative algebra over a field F and of an associative ring of order p^3 where p is a prime. These results are useful in studying the structure of the algebra and the ring.

Theorem 1 Let A be a three-dimensional associative algebra over a field F, $N(A)$ the nilpotent radical of A. If $N(A)\neq 0$, then there exists a one-dimensional ideal of A included in $N(A)$.

Proof If dim $N(A)=1$, then $N(A)$ is just desired. Now suppose dim $N(A)\neq 1$ and consider the following two cases.

(I) dim $N(A)=2$. In this case, $A/N(A)\cong F$, and hence, there is an idempotent element u in $A/N(A)$. By the idempotent lifting theorem, there exists an idempotent element e in A as a lifting of u It is not difficult to check that e is linearly independent with any non-zero element in $N(A)$. Thus, for any $a\in A$, we have

$$a=\alpha e+\beta n \quad , \qquad \alpha,\beta\in F$$

where $n\in N(A)$ and $n\neq 0$. Since $N(A)$ is the nilpotent radical, we can assert that $N(A)^2\neq N(A)$. and hence, either $N(A)^2\neq 0$ or $N(A)^2=0$. If $N(A)^2\neq 0$, we have dim $N(A)^2=1$, which shows that $N(A)^2$ is just required. Now assume $N(A)^2=0$ and prove that there exists a one-dimensional ideal of A in $N(A)$:

(1) If $A(N(A))=N(A)A=0$, then $N(A)$ is a direct summand of A. Since $N(A)^2=0$, for any non-zero element n in $N(A)$, $F[n]$ is a one-dimensional ideal of A in $N(A)$.

(2) If $A(N(A))\neq 0$ and $N(A)A=0$, then there exists an element n' in $N(A)$ such that $en'\neq 0$. Obviously, $F[en']$ is a one-dimensional ideal of A in $N(A)$.

Similarly, if $A(N(A))=0$ and $N(A)A\neq 0$, then there exists a one-dimensional ideal of A in $N(A)$.

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(3) If $A(N(A))\neq 0$ and $N(A)A\neq 0$, and $A(N(A))A\neq 0$, then there exists an element n' in $N(A)$ such that $en'e\neq 0$. Evidently, $F[en'e]$ is a one-dimensional ideal of A in $N(A)$. If $A(N(A))\neq 0$, $N(A)A\neq 0$ and $A(N(A))A=0$, then there exists an element $n'\in N(A)$ such that $en'\neq 0$, and hence $F[en']$ is a one-dimensional ideal of A in $N(A)$.

(Ⅱ) dim $N(A)=3$. In this case, $N(A)=A$, that is, A is a nilpotent algebra.

(1) If $A^2=0$, then, for any non-zero element a in A, $F[a]$ is a one-dimensional ideal of A.

(2) If $A^2\neq 0$, then from $A^2\neq A$ follows that either dim $A^2=1$ or dim $A^2=2$. Thus if dim $A^2=1$, A^2 is just a one-dimensional ideal of A; Otherwise, from $A^3\neq A^2$ follows that either $A^3=0$ or dim $A^3=1$. If $A^3=0$, then for any non-zero element $a\in A^2$, $F[a]$ is a one-dimensional ideal of A. If dim $A^3=1$, then A^3 is a one-dimensional ideal of A.

Summarizely, if $N(A)\neq 0$, then there exists a one-dimensional ideal of A inclluded in $N(A)$. This completes the proof of the theorem.

Theorem 2 Let A be a non-simple three-dimensional associative algebra over a field F. Then A has its one-dimensional proper ideal.

Proof Let $N(A)$ denote the N-radical of A. If $N(A)=0$, then A is an N-semisimple algebra. By the structure theorem for N-semisimple algebras, we have

$$A=A_1\oplus A_2\oplus\cdots\oplus A_n$$

where A_i are the simple algebra, and $n>1$. Thus A has at least one direct summand A_i as a one-dimensional ideal of A.

Now suppose $N(A)\neq 0$. By the theorem 1, there exists a one-dimensional ideal of A included in $N(A)$. This completes the proof.

Theorem 3 Let R be an associative ring of order p^3 where p is a prime number. (Ⅰ) If R is a simple ring, then R is a field; (Ⅱ) If R is a non-simple ring, then R has a proper ideal of order p.

Proof (Ⅰ) R is a simple ring. By the Wedderburn-Artin theorem, we have

$$R\cong D_n$$

where D is a division ring. Since $(D,+)$ is an subgroup of $(R,+)$ and $|D|>1$, we have

$$|D|=p^i \qquad (1\leqslant i\leqslant 3)$$

and hence

$$p^3=|R|=|D_n|=(p^i)^{n^2}=p^{in^2}$$

which implies that $n=1$. So R is a division ring, and hence R is a field.

(Ⅱ) R is a non-simple ring. We now prove that R has its proper ideal of order p.

(1) If $(R, +)$ is a cyclic group, to say, $(R, +)=\langle a\rangle$ where $o(a)=p^3$, then $\langle p^2a\rangle$ is an ideal of order p in R.

(2) If $(R, +)=\langle a\rangle\oplus\langle b\rangle$ where $o(a)=p$ and $o(b)=p^2$. Since R is a non-simple, R has a proper ideal I where $|I|=p$ or $|I|=p^2$. If $|I|=p$, our result is true. If $|I|=p^2$, and $(I, +)$ is a cyclic group, to say, $(I, +)=\langle c\rangle$ where $o(c)=p^2$, then $\langle pc\rangle$ is an ideal of order p in R. If $|I|=p^2$ and $(I, +)$ is not a cyclic group, then $(I, +)=\langle a\rangle\oplus\langle pb\rangle$ and $\langle pb\rangle$ is an ideal of order p in R.

(3) $(R, +)=\langle a\rangle\oplus\langle b\rangle\oplus\langle c\rangle$ where $o(a)=o(b)=o(c)=p$. In this case, we can regard R as three-dimensional associative algebra over the field $F(p)$. By the theorem 2, R has its proper ideal of order p. Our results follow from this.

Here we should point out, part (Ⅱ) of the theorem 3 is due to Liu Keqin [2], but our proof is different from his.

References

[1] N.Jacobson,Basic Algebra Ⅱ, W. H. Freeman and Company, San Francisco, (1974).
[2] Liu Keqin, J. of Math. (PRC), Vol.2(1982), No.1, 57—74.