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Subclasses of p-Valent Meromorphic Starlike Functions*

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Abstract Let M_{n+p-1} denote the class of functions $f(z)=\frac{1}{z^p}+\frac{a_0}{z^{p-2}}+\frac{a_1}{z^{p-2}}+\cdots+a_{n+p-1}z^n+\cdots$, regular and p-valent in the annulus $0<|z|<1$ and satisfying

$$\operatorname{Re}\left(\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}-2\right)<-\frac{n+p-1}{n+p},\quad |z|<1,\ n>-p$$

where
$$D^{n+p-1}f(z)=\frac{1}{z^p}\left(\frac{z^{n+2p-1}f(z)}{(n+p-1)!}\right)^{(n+p-1)}.$$

$M_{n+p}\subset M_{n+p-1}$ is proved. Since M_0 is the subclass of p-valent meromorphically starlike functions, all functions in M_{n+p-1} are p-valent meromorphically starlike functions. Further the integrals of functions in M_{n+p-1} are considered.

I. Introduction Let Σ_p be the class of functions

$$f(z)=\frac{1}{z^p}+\frac{a_0}{z^{p-1}}+\cdots+a_{n+p-1}z^n+\cdots\cdots \tag{1}$$

which are regular and p-valent in $E-\{0\}$ where $E=\{z:|z|<1\}$ and p a positive integer. A function f in Σ_p is said to belong to Σ_p^*, the class of p-valent meromorphically starlike functions, if and only if $\operatorname{Re} zf'(z)/f(z)<0$, $z\in E$. A function f in Σ_p is said to belong to $\Sigma_p^*(a)$, the class of p-valent meromorphically starlike functions of order a, $0\leqslant a<1$, if and only if $\operatorname{Re} zf'(z)/f(z)<-pa$, $z\in E$. The Hadamard product or convolution of f, $g\in\Sigma_p$ is denoted by $f*g$. Let

$$D^{n+p-1}f(z)=\frac{1}{z^p(1-z)^{n+p}}*f(z)=\frac{1}{z^p}\left(\frac{z^{n+2p-1}f(z)}{(n+p-1)!}\right)^{(n+p-1)}$$
$$=\frac{1}{z^p}+\frac{n+p}{z^{p-1}}a_0+\frac{(n+p)(n+p+1)}{2!\,z^{p-2}}a_1+\cdots \tag{2}$$

where n is any integer $>-p$.

In this paper we shall show that for a function $f(z)$ in Σ_p, which satisfies one of the conditions

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$$\mathrm{Re}\left(\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}-2\right)<-\frac{n+p-1}{n+p},\quad |z|<1, n>-p \tag{3}$$

is p-valently starlike in $0<|z|<1$. More precisely it is shown that for the classes M_{n+p-1} of functions in Σ_p satisfying (3)

$$M_{n+p}\subset M_{n+p-1} \tag{4}$$

holds. Since $M_0\subset\Sigma_p^*$, it follows that all members of M_{n+p-1} are p-valent meromorphically starlike functions. Further we obtain class preserving integral operators for functions in M_{n+p-1}. While obtaining these integral operators we have been able to extend some of the earlier results of S.K. Bajpai [1]. Similar problems were treated in [2] and [3]. Techniques are similar to those in [5].

2. The classes M_{n+p-1}

Theorem 1 $M_{n+p}\subset M_{n+p-1}$, $n>-p$.

Proof Let $f(z)\in M_{n+p}$ and

$$g(z)=\frac{1}{z^p}[z^pD^{n+p-1}f(z)]^{p/(n+p)} \tag{5}$$

Differentiating (5) logarithmically and using the identity

$$z(D^{n+p-1}f(z))'=(n+p)D^{n+p}f(z)-(n+2p)D^{n+p-1}f(z) \tag{6}$$

we obtain

$$u(z)=zg'(z)/g(z)=p\left(\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}-2\right) \tag{7}$$

Equation (7) may be written as

$$\frac{u(z)+2p}{p}=D^{n+p}f(z)/D^{n+p-1}f(z). \tag{8}$$

Again taking logarithmic derivative of (8) and using (6) we obtain

$$\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)}-2+\frac{n+p}{n+p+1}=\frac{1}{n+p+1}\left[-(n+p+1)+\frac{n+p}{p}(u(z)+2p)+\frac{zu'(z)}{u(z)+2p}\right].$$

In order to prove the Theorem it suffices to prove that $\mathrm{Re}\,u(z)/p<-(n+p(-1)/(n+p)$ for $|z|<1$, Define a regular function $w(z)$ in E by

$$u(z)=-p\frac{1-\left(1-2\frac{n+p-1}{n+p}\right)w(z)}{1+w(z)}=-\frac{p}{n+p}\,\frac{n+p+(n+p-2)w(z)}{1+w(z)} \tag{9}$$

clearly $w(0)=0$. From (9) we have

$$-(n+p+1)+\frac{n+p}{p}(u(z)+2p)+\frac{zu'(z)}{u(z)+2p}$$

$$=-(n+p+1)+\frac{n+p+(n+p+2)w(z)}{1+w(z)}+\frac{2zw'(z)}{(n+p+(n+p+2)w(z))(1+w(z))}.$$

We claim that $|w(z)|<1$ in $|z|<1$, for otherwise by Jack's lemma[4] there exists z_0, $|z_0|<1$ such that

$$z_0 w'(z_0) = k w(z_0) \quad \text{where} \quad k \geqslant 1.$$

Then

$$\mathrm{Re}\frac{D^{n+p+1} f(z_0)}{D^{n+p} f(z_0)} - 2 + \frac{n+p}{n+p+1}$$

$$= \mathrm{Re}\frac{1}{n+p+1}\{-(n+p+1) + \frac{n+p}{p}(u(z_0)+2p) + \frac{z_0 u'(z_0)}{u(z_0)+2p}\}$$

$$= \mathrm{Re}\frac{1}{n+p+1}\{\frac{2kw(z_0)}{(1+w(z_0))(n+p+(n+p+2)w(z_0))}\} \geqslant \frac{1}{2(n+p+1)^2} > 0 .$$

which contradicts the fact that $f(z) \in M_{n+p}$.

Hence $|w(z)| < 1$ in $|z| < 1$ and the result follows.

Theorem 2 Let p be a positive integer and n any integer greater than $-p$. If $f(z) \in M_{n+p-1}$ and $c - p + 1 > 0$ then

$$F(z) = \frac{c-p+1}{z^{c+1}} \int_0^z t^c f(t)\,dt = \frac{1}{z^p} + \frac{a_0(c-p+1)}{c-p+2} \frac{1}{z^{p-1}}$$

$$+ \frac{a_1(c-p+1)}{c-p+3} \frac{1}{z^{p-2}} + \cdots = \sum_{j=0}^{\infty} \frac{c-p+1}{c-p+1+j} \frac{1}{z^{p-j}} * f(z)$$

also belongs to M_{n+p-1}.

Proof Let $f(z) \in M_{n+p-1}$ and

$$g(z) = \frac{1}{z^p}[z^p D^{n+p-1} F(z)]^{p/(n+p)} \tag{10}$$

In order to prove the Theorem it suffices to prove that

$$\mathrm{Re}\, u(z) = \mathrm{Re}(zg'(z)/g(z)) < -p\frac{n+p-1}{n+p} .$$

Define a regular function $w(z)$ in $|z| < 1$ by

$$u(z) = p(D^{n+p}F(z)/D^{n+p-1}F(z) - 2)$$

$$= -p(\frac{n+p-1}{n+p} + \frac{1}{n+p}\frac{1-w(z)}{1+w(z)}) = -p\frac{n+p+(n+p-2)w(z)}{(n+p)(1+w(z))} \tag{11}$$

Clearly $w(0) = 0$. Since $f(z) \in M_{n+p-1}$ we have

$$\mathrm{Re}(D^{n+p}f(z)/D^{n+p-1}f(z) - 2) < -\frac{n+p-1}{n+p} . \tag{12}$$

From the defination of $F(z)$ we have the following identities:

$$z(D^{n+p-1}F(z))' = (n+p)D^{n+p}F(z) - (n+2p)D^{n+p-1}F(z) \tag{13}$$

and

$$z(D^{n+p-1}F(z))' = (c-p+1)D^{n+p-1}f(z) - (c+1)D^{n+p-1}F(z) . \tag{14}$$

Using the relations (13) and (14) the condition (12) may be written as

$$\mathrm{Re}\{\frac{(n+p+1)D^{n+p-1}F(z)/D^{n+p}F(z) - (n+2p-c)}{(n+p) - (n+2p-c-1)D^{n+p-1}F(z)/D^{n+p}F(z)} - 2\} < -\frac{n+p-1}{n+p} \tag{15}$$

we have to prove that (15) implies

$$\mathrm{Re}(D^{n+p}F(z)/D^{n+p-1}F(z)-2)<-\frac{n+p-1}{n+p}$$

Differentiating (11) logarithmically and using the identity (13) we obtain

$$\frac{(n+p+1)D^{n+p+1}F(z)/D^{n+p}F(z)-(n+2p-c)}{(n+p)-(n+2p-c-1)D^{n+p-1}F(z)/D^{n+p}F(z)}-2 \tag{16}$$

$$=-\left(\frac{n+p-1}{n+p}+\frac{1}{n+p}\frac{1-w(z)}{1+w(z)}\right)+\frac{2zw'(z)}{(n+p)(1+w(z))(c-p+1+(c-p+3)w(z))}.$$

we claim that $|w(z)|<1$. For otherwise by Jack's lemma[4] there exists z_0, $|z_0|<1$ such that

$$z_0w'(z_0)=kw(z_0),\quad k\geqslant 1 \tag{17}$$

combining (16) and (17) we obtain

$$\frac{(n+p+1)D^{n+p+1}F(z_0)/D^{n+p}F(z_0)-(n+2p-c)}{(n+p)-(n+2p-c-1)D^{n+p-1}F(z_0)/D^{n+p}F(z_0)}-2$$

$$=-\left(\frac{n+p-1}{n+p}+\frac{1}{n+p}\frac{1-w(z_0)}{1+w(z_0)}\right)+\frac{2kw(z_0)}{(n+p)(1+w(z_0))(c-p+1+(c-p+3)w(z_0))}$$

Thus the real part of left side of the above equality $\geqslant -\frac{n+p-1}{n+p}+\frac{1}{2(n+p)(c-p+2)}>-\frac{n+p-1}{n+p}$ which contradicts (12). Hence $|w(z)|<1$ and from (11) it follows that $F(z)\in M_{n+p-1}$.

Putting $n=0$, $c=p=1$ in to the statement of Theorem 2 the following result of S.K. Bajpai [1] is obtained.

Corollary 1 If $f(z)\in\Sigma^*$ then $F(z)=\frac{1}{z^2}\int_0^z t f(t)\mathrm{d}t$ also belongs to Σ^*.

Theorem 3 Let p be a positive integer and n be any inter $>-p$. If

$$\mathrm{Re}(D^{n+p}f(z)/D^{n+p-1}f(z)-2)<\frac{1-2(n+p-1)(c-p+2)}{2(n+p)(c-p\ \ 2)},\ |z|<1.$$

and $c-p+1>0$, then $F(z)=\frac{c-p+1}{z^{c+1}}\int_0^z t^c f(t)\mathrm{d}t$ belongs to M_{n+p-1}.

Proof is similar to that of Theorem 2.

Putting $n=0$, $p=c=1$ into the statement of Theorem 3 we obtain

Corollary 2 If $\mathrm{Re}\, zf'(z)/f(z)<1/4$ for $|z|<1$ and $F(z)=\frac{1}{z^2}\int_0^z tf(t)\mathrm{d}t$, then $F(z)\in\Sigma^*$

Corollary 2 is an extension of corollary 1.

Theorem 4 If $n>-p$ and $f(z)\in M_{n+p-1}$ then the function $F(z)$ defined by

$$F(z)=\frac{n+p}{z^{n+2p}}\int_0^z t^{n+2p-1}f(t)\mathrm{d}t$$

belongs to M_{n+p}.

Proof From the definition of $F(z)$ we have

$$D^{n+p-1}f(z)=D^{n+p}F(z)$$

and

$$(n+p)D^{n+p}f(z)=(n+p+1)D^{n+p+1}F(z)-D^{n+p}F(z).$$

From these relations and the fact that $f(z)\in M_{n+p-1}$, we get

$$\mathrm{Re}\left(\frac{(n+p+1)D^{n+p+1}F(z)-D^{n+p}F(z)}{(n+p)D^{n+p}F(z)}-2\right)$$

$$=\mathrm{Re}\left(\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}-2\right)<-\frac{n+p-1}{n+p}$$

from which it follows that

$$\mathrm{Re}\left(\frac{D^{n+p+1}F(z)}{D^{n+p}F(z)}-2\right)<-\frac{n+p}{n+p+1}$$

Thus $F(z)\in M_{n+p}$.

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condition in a neighborhood of x^*,

$$\|H-H^*\|\leq K\|x-x^*\|,\quad \forall H\in\partial^2 f(x),\quad H^*=\partial^2 f(x^*),$$

then the algorithm given above possesses the convergency of order 2.

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