Journal of Mathematical Reaserch and Exposition
Vol.9, No.3, Aug., 1989

Generlization of Direct Product and Generation of Sets of Orthogonal Functions

Wang Zhongde

(Beijing University of Posts & Telecommunications)

Abstract

Two new kinds of direct product of matrices are defined. Their properties are investigated. Direct products of matrix and set of continuous functions are also defined. Many complete sets of orthogonal functions, such as those sets given by Walsh [2], Paley [3], Chrestenson [4], and Watari [5], may be generated by these new kinds of direct product. Direct products are also applicable to the generation of sets of piecewise orthogonal functions.

0. Introduction

The direct product (or Kronecker product)[1] is an important matrix manipulation. The direct product of two matrices $A(m_0\times n_0)$ and $B(m_1\times n_1)$ (The number in a pair of brackets behind a matrix symbol indicates the dimensions of the matrix), represented by $A\otimes B$, is a matrix of dimension $(m_0m_1\times n_0n_1)$. Its element on the ith row and jth columm is given by

$$(A\otimes B)_{ij}=a_{i_0j_0}b_{i_1j_1}$$

where a and b are elements of A and B respectively.

$$\begin{cases} i=i_0m_1+i_1 \\ i_0=[\frac{i}{m_1}]\in Z_{m_0}:=\{0,1,\cdots,m_0-1\};\ [a]=\max n,\ n\in Z,\ n\leqslant a \\ i_1=i \bmod m_1\in Z_{m_1} \end{cases} \qquad (1)$$

$$j=j_0n_1+j_1$$

$$j_0=[\frac{j}{n_1}]\in Z_{n_0}$$

$$j_1=j \bmod n_1\in Z_{n_1}$$

Given i and j, one may uniquely obtain i_0, i_1, j_0, j_1, or vice versa.

The following are some properties of the direct product. Their proofs can be found in any textbook on matrix analysis.

(1) $(A+B)\otimes(C+D)=A\otimes C+B\otimes C+A\otimes D+B\otimes D$

* Received Dec.22, 1987.

(2) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

(3) $(AC) \otimes (BD) = (A \otimes B)(C \otimes D)$

(4) The direct product of two orthogonal matricas is orthogonal.

(5) The direct product of two symmetric matrices is symmetric. In addition, the self direct product, or direct power of A is represented by

$$A^{(2)} = A \otimes A, \quad A^{(n)} = A^{(n-1)} \otimes A = A \otimes A^{(n-1)}$$

I. The Direct Product of the Second Kind

Representing rows of B by $B_0, B_1, \cdots, B_{n_1-1}$, we define the direct product of the second kind of A and B by

$$A \bigcirc\!\!\!\!\wedge\, B = \begin{pmatrix} A \otimes B_0 \\ A \otimes B_1 \\ \vdots \\ A \otimes B_{n_1-1} \end{pmatrix} \tag{2}$$

where $\otimes$ represents the ordinary direct product. We sometimes call it the direct product of the first kind for distinction. Obviously, $A \bigcirc\!\!\!\!\wedge\, B$ may be obtaind from $A \otimes B$ by a row permutation,

$$A \bigcirc\!\!\!\!\wedge\, B = P(A \otimes B).$$

where P is a permutation matrix of order $m_0 m_1$. Its elements are

$$p_{ij} = \delta_{i \bmod m_0 [\frac{j}{n_1}]} \delta_{[\frac{i}{n_0}] j \bmod n_i}$$

where $\delta_{kl} = 1$ for $k = 1$, otherwise $\delta_{kl} = 0$. Let

$$\begin{aligned} i &= i_1 m_0 + i_0 \\ i_0 &= i \bmod m_0 \in Z_{m_0} \\ i_1 &= [\frac{i}{m_0}] \in Z_{m_1} \end{aligned} \tag{3}$$

Representing i by two digits $i_0 i_1$, the ith row of $A \bigcirc\!\!\!\!\wedge\, B$ may then be represented by

$$(A \bigcirc\!\!\!\!\wedge\, B)_{i_1 i_0} = A_{i_0} \otimes B_{i_1} \tag{4}$$

Following theorems are concerning the properties of the direct product of the second kind.

Theorem 1

$$(A+B) \bigcirc\!\!\!\!\wedge\, (C+D) = A \bigcirc\!\!\!\!\wedge\, C + B \bigcirc\!\!\!\!\wedge\, C + A \bigcirc\!\!\!\!\wedge\, D + B \bigcirc\!\!\!\!\wedge\, D \tag{5}$$

Proof

$$\begin{aligned} (A+B) \bigcirc\!\!\!\!\wedge\, (C+D) &= P[(A+B) \otimes (C+D)] \\ &= P[A \otimes C + B \otimes C + A \otimes D + B \otimes D] \\ &= A \bigcirc\!\!\!\!\wedge\, C + B \bigcirc\!\!\!\!\wedge\, C + A \bigcirc\!\!\!\!\wedge\, D + B \bigcirc\!\!\!\!\wedge\, D \end{aligned}$$

Theorem 2

$$(A \circledwedge B) \circledwedge C = A \circledwedge (B \circledwedge C) \tag{6}$$

Proof The difference between the direct product of the second and the first kinds is just on the ordering of the rows. Therefore, we need only to prove that the ith rows of both sides are the same.

Suppose the numbers of row dimensions of A, B, and C are m_0, m_1, and m_2, respectively. On the left hand side of (6), the i_0th row of A and i_1th row of B generate the $(i_1 m_0 + i_0)$th row of $A \circledwedge B$. The $(i_1 m_0 + i_0)$th row of $A \circledwedge B$ and the i_2th row of C generate $i = (i_2 m_1 m_0 + i_1 m_0 + i_0)$th row of $(A \circledwedge B) \circledwedge C$. On the other hand, on the right hand side of (6), the i_1th row of B and the i_2th row of C generate the $(i_2 m_1 + i_1)$th row of $B \circledwedge C$. The i_0th row of A and the $(i_2 m_1 + i_1)$th row of $B \circledwedge C$ generate the $(i_2 m_1 + i_1) m_0 + i_0 = i$th row of $A \circledwedge (B \circledwedge C)$. The same rows of A, B, and C generate the same row on both sides of the equation. Therefore, the equation is valid.

This theorem allows us to represent the power of the direct product of the second kind by

$$A^{(\hat{2})} = A \circledwedge A; \quad A^{(\hat{n})} = A^{(\widehat{n-1})} \circledwedge A = A \circledwedge A^{(\widehat{n-1})} . \tag{7}$$

Theorem 3 The power of the direct product of the second kind of a symmetric matrix is symmetric.

Proof We use induction to prove this theorem. Let A be a symmetric matrix. A symmetric matrix must be square. Let the order of A be m. The elements of $A^{(\hat{2})}$ are

$$a(2)_{ij} = a_{i \bmod m [\frac{j}{m}]} \, a_{[\frac{i}{m}] j \bmod m} .$$

On the other hand,

$$a(2)_{ji} = a_{j \bmod m [\frac{j}{m}]} \, a_{[\frac{j}{m}] i \bmod m} .$$

Since A is symmetric

$$a_{i \bmod m [\frac{j}{m}]} = a_{[\frac{j}{m}] i \bmod m} .$$

Hence, $a(2)_{ij} = a(2)_{ji}$, and $A^{(\hat{2})}$ is symmetric. Suppose $A^{(\hat{n})}$ is symmetric. Write $A^{(\widehat{n+1})} = A^{(\hat{n})} \circledwedge A$. Its element on the ith row and jth column is given by

$$a(n+1)_{ij} = a(n)_{i \bmod m^n [\frac{j}{m}]} \, a_{[\frac{i}{m^n}] j \bmod m} .$$

On the other hand, write $A^{(\widehat{n+1})} = A \circledwedge A^{(\hat{n})}$. Its element on the jth row and ith column is given by

$$a(n+1)_{ji} = a(n)_{j \bmod m [\frac{i}{m^n}]} \, a_{[\frac{j}{m}] i \bmod m^n} .$$

Since both A and $A^{(\hat{n})}$ are symmetric,

$$a_{j \bmod m [\frac{i}{m^n}]} = a_{[\frac{i}{m^n}] j \bmod m} , \qquad a(n)_{[\frac{j}{m}] i \bmod m^n} = a_{i \bmod m^n [\frac{j}{m}]} .$$

Therefore, $a(n+1)_{ij} = a(n+1)_{ji}$. Thus $A^{(\widehat{n+1})}$ is symmetric, and the theorem is proven.

Theorem 4

$$(AC) \circledwedge (BD) = (A \circledwedge B)(C \otimes D) \tag{8}$$

Proof If P is a permutation matrix which contains precisely a single 1 in each row and each column, and zeros elsewhere, one may easily to verify that

$$(PA)B = P(AB) \tag{9}$$

where A and B are two matrices with appropriate dimensions. Hence

$$(A \bigcirc C)(B \otimes D) = [P(A \otimes C)](B \otimes D)$$
$$= P[(A \otimes C)(B \otimes D)] = P[(AC) \otimes (BD)] = (AC) \bigcirc (BD)$$

Since the direct product of the second kind is a permuted alternative of the direct product of the first kind, and the orthogonality of a matrix does not change by permutation, we have

Collolary 1 The direct product of the second kind of two orthogonal matrices is orthogonal.

3. The Direct Product of the Third Kind

The direct product of the third kind of two matrices $A(m_0 \times n_0)$ and $B(m_1 \times n_1)$ is defined as

$$A \oslash B = \begin{pmatrix} \bar{A} \otimes B_0 \\ (\bar{I}A) \otimes B_1 \\ \vdots \\ (\bar{I}^k A) \otimes B_k \\ \vdots \\ (\bar{I}^{m_1-1}\bar{A}) \otimes B_{m_1-1} \end{pmatrix} \tag{10}$$

where $\bar{I}$ is the opposite diagonal unit matrix of order m. Its elements are zeros except those along the diagonal from north-east to south-west where they are '1's. $B_k (k \in Z_m)$ is the kth row of B. Obviously, $A \oslash B$ is another permuted alternative of $A \otimes B$. Represent i by two digits $i_1 i_0$, and let

$$i_0' = \begin{cases} i_0 & \text{for } i_1 \text{ even} \\ \bar{i}_0 = m_0 - 1 - i_0 & \text{for } i_1 \text{ odd} \end{cases} \tag{11}$$

where $\bar{i}_0$ is the complement code of i_0. Then the ith row of $A \oslash B$ can be generated by

$$(A \oslash B)_{i_1 i_0} = A_{i_0'} \otimes B_{i_1} \tag{12}$$

Since the direct product of the third kind is another permuted version of the direct product, similar to the theorem 1,4 and the collolary 1, we have

Theorem 5

$$(A+B) \oslash (C+D) = A \oslash C + B \oslash C + A \oslash D + B \oslash D \tag{13}$$

Theorem 6

$$(AC) \oslash (BD) = (A \oslash B)(C \otimes D) \tag{14}$$

and

Collolary 2 The direct product of the third kind of two orthogonal matrices is orthogonal.

Now we shall show that the associative law holds for the direct product of the third kind too.

Theorem 7

$$(A \oslash B) \oslash C = A \oslash (B \oslash C) \tag{15}$$

Proof Similar to the case for the direct product of the second kind, we need only to show that the ith rows of both sides are identical. Represent i by three digits $i_2 i_1 i_0 (i_k \in m_k,\ k=0,1,2)$, where m_0, m_1, and m_2 are the numbers of row dimension of A, B, and C respectively. The ith row on the left of (15) is

$$[(A \oslash B) \oslash C]_{i_2 i_1 i_0} = (A \oslash B)_{(i_1 i_0)'} \otimes C_{i_2}$$

When i_2 is even

$$(A \oslash B)_{(i_1 i_0)'} = (A \oslash B)_{i_1 i_0} = A_{i_0'} \otimes B_{i_1}$$
$$= \begin{cases} A_{i_0} \otimes B_{i_1} & \text{for } i_1 \text{ even} \\ A_{\bar{i}_0} \otimes B_{i_1} & \text{for } i_1 \text{ odd}. \end{cases}$$

When i_2 is odd

$$(A \oslash B)_{(i_1 i_0)'} = (A \oslash B)_{\overline{i_1 i_0}} = (A \oslash B)_{\bar{i}_1 \bar{i}_0} = A_{\bar{i}_0'} \otimes B_{\bar{i}_1}$$
$$= \begin{cases} A_{\bar{i}_0} \otimes B_{\bar{i}_1} & \text{for } \bar{i}_1 \text{ even} \\ A_{i_0} \otimes B_{\bar{i}_1} & \text{for } \bar{i}_1 \text{ odd}. \end{cases}$$

On the other hand, the ith row of the right hand side of (15) is

$$[A \oslash (B \oslash C)]_{i_2 i_1 i_0} = A_{i_0'} \otimes (B \oslash C)_{i_2 i_1}$$

where

$$i_0' = \begin{cases} i_0 & \text{for } i_2 i_1 \text{ even} \\ \bar{i}_0 = m_0 - 1 - i_0 & \text{for } i_2 i_1 \text{ odd}. \end{cases}$$

However

$$(B \oslash C)_{i_2 i_1} = B_{i_1'} \otimes C_{i_2} = \begin{cases} B_{i_1} \otimes C_{i_2} & \text{for } i_2 \text{ even} \\ B_{\bar{i}_1} \otimes C_{i_2} & \text{for } i_2 \text{ odd} \end{cases}$$

Representing two digits $i_2 i_1$ by $i_2 m_1 + i_1$, one may judge that there are three cases for $i_2 i_1$ even: (1) Both i_2 and i_1 are even; (2) i_2 is odd, but both m_1 and i_1 are even; and (3) i_2, m_1 and i_1 are all odd. However, the last two cases is equivalent to say that both i_2 and $\bar{i}_1 = m_1 - 1 - i_1$ are odd.

There are three cases for $i_2 i_1$ odd: (1) i_2 is even, but i_1 is odd; (2) Both i_2 and m_1 are odd, but i_1 is even; and (3) Both i_2 and i_1 are odd, but m_1 is even. However, the last two cases is equivalent to say that i_2 is odd, but $\bar{i}_1$ is even.

From the above arguements one may see that in all cases the ith rows of both sides of equation (15) are generated by the same rows of A, B, and C. Therefore the equation is valid.

This theorem allows us to represent the power of the direct product of the third kind by

$$A^{(\overset{\vee}{2})} = A \bigcirc\!\!\!\!\vee A; \quad A^{(\overset{\vee}{n})} \equiv A^{(n \overset{\vee}{-} 1)} \bigcirc\!\!\!\!\vee A = A \bigcirc\!\!\!\!\vee A^{(n \overset{\vee}{-} 1)} \tag{16}$$

Now we shall show a nice property of the sign changes of the direct product of the third kind. We call the number of sign changes within the ith row of a matrix A the sequency of that row, and represent it by $\mu(A)_i$ or μ_i. If $\mu_{i+1} > \mu_i$ holds for any i, we say that the matrix is sequency ordered. For a $m \times m$ square matrix, the maximal sequency is $m-1$. If it is sequency ordered, one must have $\mu_i = i$, $(i \in Z_m)$.

Theorem 8 If A contains no null element, and both $A(m_0 \times m_0)$ and $B(m_1 \times m_1)$ are sequency ordered, then $A \bigcirc\!\!\!\!\vee B$ is sequency ordered.

Proof Let us consider μ_i, the sequency of the ith row of $A \bigcirc\!\!\!\!\vee B$. Representing i by two digits $i_1 i_0$, we discuss the problem in two cases.

(1) i_1 is even. As is shown by (10), (11), and (12), $(A \bigcirc\!\!\!\!\vee B)_{i_1 i_0} = A_{i_0} \otimes B_{i_1}$. Since i_1 is even, the signs of the first and the last elements of B_{i_1} is identical. Since A contains no null element, each element of A will make $\mu(B)_{i_1}$ sign changes for $A_{i_0} \otimes B_{i_1}$. The total sign changes in $A_{i_0} \otimes B_{i_1}$ are the sign changes made by m elements of A_{i_0} plus the sign changes of A_{i_0} itself. Hence,

$$\mu_i = \mu_{i_1 i_0} = \mu(B)_{i_1} m_0 + \mu(A)_{i_0} = i_1 m_0 + i_0 = i$$

(2) i_1 is odd. From (10) thru (12), $(A \bigcirc\!\!\!\!\vee B)_{i_1 i_0} = A_{\bar{i}_0} \otimes B_{i_1}$, where $\bar{i}_0$ is given by (11). Since i_1 is odd, the sign of the last element of B_{i_1} is opposite to that of the first element. Each element of $A_{\bar{i}_0}$ shall make $\mu(B)_{i_1} + 1$ sign changes if the next element of $A_{\bar{i}_0}$ is of the same sign as the first one, or only make $\mu(B)_{i_1}$ sign changes if the next element of $A_{\bar{i}_0}$ is of the opposite sign to the first one. The last element of $A_{\bar{i}_0}$ will only make $\mu(B)_{i_1}$ sign changes. The total sign changes will be $\mu_i = \mu_{i_1 i_0} = [\mu(B)_{i_1} + 1] m_0 - \mu(A)_{\bar{i}_0} - 1 = (i_1 + 1) m_0 - \bar{i}_0 - 1 = i_1 m_0 + i_0 = i$ Hennce, $\mu_i = i$ holds for both cases. Therefore, $A \bigcirc\!\!\!\!\vee B$ is sequency ordered.

4. Applications

Let A be an orthogonal matrix of order m, and $AA^T = m\mathrm{I}$. T represents the transposition. I is the identity matrix. We define m step functions on $[0,1)$ as

$$\varphi_i(x) = a_{ij} \quad j/m \leqslant x < (j+1)/m \quad j \in Z_m;\ i \in Z_m$$

where a_{ij} are elements of A. Since

$$\int_0^1 \varphi_i(x) \varphi_k(x) \mathrm{d}x = \sum_{j=0}^{m-1} a_{ij} a_{kj} = \delta_{ik},$$

we see that $\{\varphi_i(x)\}$ $(i \in Z_m)$ is an orthogonal basis in R^m. $\{\varphi_i(x)\}$ will be referred to as the orthogonal function set associated with A; or inversely, we say that A is associated with $\{\varphi_i(x)\}$. The main application of the direct product of the second and third kinds is to generate orthogonal matrices of higher order from that of lower order. Since every orthogonal matrix is associated with a set of step orthogonal functions, the direct products of the second and third kinds are applicable to the generation of step orthogonal function sets. Following are some examples.

(1) Let

$$H=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix};\quad G=[e^{-2\pi ikl/p}];\quad G_j=[e^{-2\pi ik_jl_j/p_j}];$$

where $i=\sqrt{-1}$; $k,l\in Z_p$; $k_j,l_j\in Z_{p_j}$; $j=0,1,\cdots$. The orthogonal function set associated with $W=\lim\limits_{n\to\infty}H^{(\check{n})}$ is the function system defined by Walsh [2], and that associated with $P=\lim\limits_{n\to\infty}H^{(\hat{n})}$ is the function system defined by Paley [3]. The set of functions associated with $CH=\lim\limits_{n\to\infty}G^{(\hat{n})}$ is the function system defined by Chrestenson [4]. The set of functions associated with $WR=G_0\bigotimes G_1\bigotimes G_2\cdots$, is the function system defined by Watari [5].

(2) Let $A_0, A_1,\cdots$, be orthogonal matrices of orders $m_0, m_1,\cdots$, respectively. The elements on the first rows of these matrices are all ones. The functions associated with A_k are represented by $\varphi_{k,i_k}(x)$, $i_k\in Z_{m_k}$. Let $p_k=\prod\limits_{i=0}^{k-1}m_i$, and represent a natural number n by

$$n=i_{k-1}p_{k-1}+i_{k-2}p_{k-2}+\cdots+i_1p_1+i_0\quad i_k\in Z_{m_k}.$$

We define the generalized Walsh function

$$\psi_n(x)=\varphi_{k,i_k}(x)\varphi_{k-1,i_{k-1}}(x)\cdots\varphi_{1,i_1}(x)\varphi_{0,i_0}(x)$$

Following Watari [5], one may prove that $\psi_n(x)$ is a complete orthogonal function system on L(0,1) which is associated with $A_0\bigotimes A_1\bigotimes\cdots$. The Watari system is a special case of this system. A sequency ordered alternative of this system may be readily obtained by using the direct product of the third kind if all A_k are sequency ordered and contain no null element.

5. Generalization of Direct Product to the Continuous Functions

Let $A(m_0\times n)$ be a matrix. $a_{i,j}$ is its element on the ith row and jth column. Let $f(x)$ be a function defined on $[0,1)$. The following m_0 functions on $[0,1)$

$$\varphi_i(x)=a_{ij}f(nx-j),\quad j/n\leqslant x<(j+1)/n,\quad j\in Z_n,\ i\in Z_{m_0}\tag{17}$$

are defined as the direct product of A and $f(x)$, and will be represented by $\{\varphi\}=A\otimes f$. Let $\{f\}:=\{f_0(x), f_1(x),\cdots, f_{m_1-1}(x)\}$ be a set of m_1 functions. Let i, i_0, and i_1 be given by (1), then the following m_0m_1 functions

$$\varphi_i(x)=a_{i_0j}f_{i_1}(nx-j),\quad j/n\leqslant x<(j+1)/n,\quad j\in Z_n,\quad i\in Z_{m_0m_1}\tag{18}$$

are definet as the direct product of A and $\{f\}$, and will be represented by $\{\varphi\}=A\otimes\{f\}$. If i, i_0, i_1 and i_0' are given by (3) and (11), then the m_0m_1 functions given by (18) are defined as the direct product of the second kind of A and $\{f\}$, and will be represented by $\{\varphi\}=A\bigcirc\{f\}$. On the other hand, the following

$m_0 m_1$ functions

$$\psi_i(x) = a_{i_0' j} f_{i_1}(nx - j), \quad j/n < x < (j+1)n, \quad j \in Z_n; \; i \in Z_{m_0 m_1} \tag{19}$$

are defined as the direct product of the third kind of A and $\{f\}$, and will be represented by $\{\psi\} = A \oslash (f)$. Obviously, $A \otimes \{f\}$, $A \oslash \{f\}$, and $A \oslash \{f\}$ define the same set of functions, but in different orderings. However, if $\{f\}$ is a set containing infinite number of functions, $A \otimes \{f\}$ loses its meaning. But $A \oslash \{f\}$ and $A \oslash \{f\}$ still have meaning. They represent the same set of infinite number of functions with different ordering.

Functions generated by direct product possess piecewise property. At a point of discontinuity, let the function be continuous to the right. Following theorems are concerning the generation of sets of orthogonal functions.

Theorem 9 If A is a orthogonal matrix of order m, $AA^T = m\mathrm{I}$, and $f(x)$ is a function on $[0, 1)$ such that $\int_0^1 f(x)^2 \mathrm{d}x = 1$, Then $A \otimes f(x)$ is a set of m orthogonal functions.

Proof According to the definition, $A \otimes f(x)$ defines a set of m functions. From (17),

$$\int_0^1 \varphi_i(x)\varphi_k(x)\mathrm{d}x = \sum_{j=0}^{m-1} a_{ij} a_{kj} \int_{j/m}^{(j+1)/m} f(mx - j)^2 \mathrm{d}x$$

$$= \frac{1}{m} \sum_{j=0}^{m-1} a_{ij} a_{kj} \int_0^1 f(y)^2 \mathrm{d}y = \delta_{ik}$$

The proof is completed.

Theorem 10 If A is an orthegonal matrix of order m, $AA^T = m\mathrm{I}$, and $\{f\}$ is a set of p orthogonal functions on $[0, 1]$, then $A \otimes \{f\}$, $A \oslash \{d\}$, and $A \oslash \{f\}$ are sets of mp orthozonal functions on $[0, 1)$.

Proof Since three kinds of direct product define the same set of functions, we need only to prove this theorem for one case. According to (3) and (18),

$$\int_0^1 \varphi_i(x)\varphi_k(x)\mathrm{d}x = \sum_{j=0}^{m-1} a_{i_0 j} a_{k_0 j} \int_{j/m}^{(j+1)/m} f_{i_1}(mx - j) f_{k_1}(mx - j)\mathrm{d}x$$

$$= \frac{1}{m} \sum_{j=0}^{m-1} a_{i_0 j} a_{k_0 j} \int_0^1 f_{i_1}(y) f_{k_1}(y)\mathrm{d}y.$$

Since $\{f\}$ is a set of orthogonal functions, $\int_0^1 f_{i_1}(y) f_{k_1}(y)\mathrm{d}y = \delta_{i_1 k_1}$. On the other hand, since $AA^T = m\mathrm{I}$, we have $\frac{1}{m}\sum_{j=0}^{m-1} a_{i_0 j} a_{k_0 j} = \delta_{i_0 k_0}$. Therefore

$$\int_0^1 \varphi_i(x)\varphi_k(x)\mathrm{d}x = \delta_{i_0 k_0}\delta_{i_1 k_1} = \begin{cases} 1 & i_0 = k_0 \text{ and } i_1 = k_1 \\ 0 & \text{otherwise} \end{cases}$$

However, the condition $i_0 = k_0$ and $i_1 = k_1$ is equivalent to that $i = k$. Thus the theorem is proven.

Let W be the Walsh matrix of order $N=2^n$([6]), $\{\lambda\}:=\{\lambda_0,\lambda_1,\cdots,\lambda_p\}=\sqrt{2k+1}\,P_k(x)$, $P_k(x)$ is the kth Legendre polynomial defined on $[0,1)$. Then $W\otimes\{\lambda\}$ is the set of piecewise orthogonal polynomials defined by Qi and Feng[2].

Theorem 11 If A is an orthogonal matrix, $AA^T=m\mathrm{I}$, and $\{f\}$ is a complete set of orthogonal functions on $[0,1)$, then $A\bigotimes\{f\}$ and $A\bigotimes\{f\}$ are complete sets of orthogonal functions on $[0,1)$.

Proof We need only to prove this theorem for one case. the orthogonality of $A\bigotimes\{f\}$ may be proven by the same procedure as the last theorem, and will be omitted here. We will prove the completeness of $A\bigotimes\{f\}$ only.

Suppose $g(x)\not\equiv 0$, and $g(x)$ is orthogonal to all functions of $\{\varphi\}=A\bigotimes\{f\}$. In other words, $\int_0^1 g(x)\varphi_i(x)\mathrm{d}x=0$ for any i. Representing i by (3), we obtain

$$\int_0^1 g(x)\varphi_i(x)\mathrm{d}x=\sum_{j=0}^{m-1}a_{i_0 j}\int_{j/m}^{(j+1)/m}g(x)f_{i_1}(mx-j)\mathrm{d}x$$

$$=\frac{1}{m}\sum_{j=0}^{m-1}a_{i_0 j}\int_0^1 g\left(\frac{y+j}{m}\right)f_{i_1}(y)\mathrm{d}y=0$$

Let $b_{i_1 j}=\int_0^1 g\left(\frac{y+j}{m}\right)f_{i_1}(y)\mathrm{d}y,\ j\in Z_m$, we get

$$\sum_{j=0}^{m-1}a_{i_0 j}b_{i_1 j}=0\,. \tag{20}$$

Since m rows of A can be viewed as a complete orthogonal basis on R^m, (20) shows that the project of the vector $b_{i_1}=[b_{i_1 0}, b_{i_1 1},\cdots,b_{i_1 m-1}]^T$ on any axis of R^m is 0. Therefore, we must have

$$b_{i_1 j}=\int_0^1 g\left(\frac{y+j}{m}\right)f_{i_1}(y)\mathrm{d}y\equiv 0.$$

The completeness of $\{f\}$ ensures that $g(x)\equiv 0$. The theorem is thus proven.

The above theorems show that the direct product can be used to generate sets of orthogonal functions with higher number of dimensions from that with lower number of dimensions, or generate new complete sets of orthogonal functions from known complete sets of orthogonal functions. The functions so generated possess piecewise property. The sets of orthogonal functions so generated are sets of piecewise orthogonal functions. Following the proof of the theorem 8, reader will have no difficulty to prove the following theorem.

Theorem 12 If A is a sequency ordered matrix containing no null element, and if $\{f\}$ is a sequency ordered set of functions, $A\bigotimes\{f\}$ is a scqueuey ordered set of functions.

Most known sets of orthogonal functions on $[0,1)$, such as the Legendre polynomials, the Walsh functions, $\{\cos m\pi x\}$, $\{\sin(m+1)\pi x\}$, $\{\cos(m+\frac{1}{2})\pi x\}$,

$\{\sin(n+\frac{1}{2})\pi x\}$, etc, are sequency ordered. The sequency ordering seems to be a natural way of ordering functions in a set of orthogonal functions. For this reason, the direct product of the third kind is more attractive when one generates sets of orthogonal functions by direct product.

References

[1] S.Bornett, Matrix Metrods for Engineers and Scientists, McGraw-Hill Book Company (UK) Limited, 1979.

[2] J.L. Walsh, Amer. J. Math. vol. 45, pp. 5—24, 1923.

[3] R.E.A.C. Paley, Proc. London Math. Sco. Vol. 34, pp. 241—279, 1932.

[4] H.F. Chrestenson, Pasific J. Math. vol. 5, pp. 17—31, 1955.

[5] C. Watari, Tohoku Math. J. vol. 10, pp. 211—241, 1958.

[6] Zhen Weixing *et al*, Theory and Applications of the Walsh Functions, Acadamy Press of Shanghhai, 1983. (In Chinese).

[7] D. Qi and Y. Feng, J. China Univ. Science & Tech., Suupplement, pp. 10—1?, June 1983. (In Chinese).