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The Generalized Noncentral Wishart Distribution*

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Abstract

In this paper, we have definited the generalized noncentral Wishart distribution for elliptically contoured distributions. Under some conditions, we have obtained its probability density function, the density function of its latent roots, the moments of the generalized variance, and the characteristic function in the series forms.

I. Introduotion

Along with the pushing of the theory of multivariate elliptically cnntoured distributions, the generalized multivariate statistic analysis is gaining a big development. But the generalized noncentral Wishart distribution takes a role in generalized multivariate statistic analysis as important as noncentral Wishart distribution does in multivariate statistic analysis. In this essay, we discuss generalized noncentral Wishart distribution under multivariate elliptically contoured distributions.

Assume Z is $n\times m$ random matrix, has an elliptically contoured distribution of $Z\sim \mathrm{LEC}_{n\times m}(M,\Sigma,\varphi)$, $\Sigma>0$. Its density funotion is:

$$(\det\Sigma)^{1/2}g[\mathrm{tr}((Z-M)\Sigma^{-1}(Z-M)')] \tag{1}$$

where M is a $n\times m$ real matrix, Σ is an $m\times m$ positive matrix, g is a certain propriate function. We call that $A=Z'Z$ is the generalized noncentral Wishart matrix. Its distribution function is denoted by $GW_m(n,\Sigma,\Omega;g)$, where $\Omega=\Sigma^{-1/2}M'M\Sigma^{-1/2}$ is noncentral parameters. When $m=1$, $\Sigma=I$, we call that A is generalized noncentral χ^2-distribution denoted by $A\sim G\chi_n^2(\delta^2;g)$. $\delta^2=M'M$. When $M=0$, $\Omega=0$, we say that A is the generalized Wishart matrix denoted by $A\sim GW_m(n,\Sigma;g)$.

Anderson & Fang[1] have given the density function of generalized Wishart distribution $GW_m(n,\Sigma;g)$. Fan[4] gave the density function of generalized noncentral Wishart distribution $GW_m(n,\Sigma,\Omega;g)$ in integral form. But density func-

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tion in this from is not appliable. We will give in this essay the density function in series form under the condition that g is Teylor expansive.

2. Density Function

We have the notes the same as in [7]. Assume k is an interger. k's partition is denoted by $\kappa=(k_1, k_2, \cdots)$, where $\Sigma_i k_i = k$, $k_1 \geqslant k_2 \geqslant \cdots$, $k_1, k_2, \cdots$, are non-negatives. Y is a symmetric matrix. Y's zonal polynomial with κ is denoted $C_\kappa(Y)$.

Lemma 1 Assume X is an $m\times n$ matrix $(n\geqslant m)$, $H=[H_1 : H_2]\in O(n)$, $H_1=(H_1)_{n\times m}$, then

$$\int_{O(n)}[\mathrm{tr}(XH_1)]^k(\mathrm{d}H)=\begin{cases}0 & k=2i+1\\ \sum_\kappa \dfrac{C_\kappa(\frac{1}{4}XX')}{(\frac{1}{2}n)_\kappa}\cdot 2^i(2i-1)!! & k=2i\end{cases} \quad (2)$$

where $(\mathrm{d}H)$ denotes the invariant measure over the Stiefel manifold $O(n)=\{H_{n\times n}: H'H=\mathrm{I}_n\}$, $i=0,1,2,\cdots$, The right of (2) is the sum of all i's partitions $\kappa=(k_1, k_2, \cdots, k_m)$ that have no more than m parts. When $\kappa=(k_1, k_2, \cdots, k_m)$,

$$(\tfrac{1}{2}n)_\kappa=\pi^{\frac{1}{4}m(m-1)}[\Gamma_m(\tfrac{1}{2}n)]^{-1}\cdot\prod_{j=1}^{m}\Gamma(\tfrac{1}{2}n+k_j-\tfrac{1}{2}(j-1)).$$

Proof When $m=n$, according to the (22), (46) in [6], we can gain the conclusion. When $m<n$, we expand X to a $n\times n$ matrix $\begin{bmatrix}X\\0\end{bmatrix}$, expand H_1 to H, then we will have the proof.

Theorem 1 Assume g is Taylor expansible on $(-\infty, +\infty)$, then we can denote the density function of the generalized noncentral Wishart matrix A as

$$\frac{\pi^{mn/2}}{\Gamma_m(\frac{1}{2}n)}(\det\Sigma)^{-\frac{n}{2}}(\det A)^{(n-m-1)/2}\sum_{k=0}^{\infty}\frac{g^{(2k)}[\mathrm{tr}(\Sigma^{-1}A+\Omega)]}{k!}\cdot$$
$$\sum_\kappa\frac{C_\kappa(\Omega\Sigma^{-\frac{1}{2}}A\Sigma^{-\frac{1}{2}})}{(\frac{1}{2}n)_\kappa} \quad (3)$$

where $A>0$, κ is k's partitions that have no more than m parts.

Proof Let $Z=H_1T$, T is an upper triangle matrix, then $A=Z'Z=T'T$, also

$$(\mathrm{d}Z)=2^{-m}[\det(Z'Z)]^{\frac{1}{2}(n-m-1)}(\mathrm{d}A)(H_1'\mathrm{d}H_1).$$

For the density function of Z is (1), then (1) changes into

$$2^{-m}(\det\Sigma)^{-\frac{1}{2}n}g[\mathrm{tr}(\Sigma^{-1}Z'Z+\Omega-2\Sigma^{-1}M'H_1T)][\det(Z'Z)]^{\frac{1}{2}(n-m-1)}(\mathrm{d}A)(H_1'\mathrm{d}H_1)$$
$$=2^{-m}(\det\Sigma)^{-\frac{1}{2}n}(\det A)^{\frac{1}{2}(n-m-1)}g[\mathrm{tr}(\Sigma^{-1}A+\Omega-2\Sigma^{-1}M'H_1T)](\mathrm{d}A)(H_1'\mathrm{d}H_1)$$

$$= 2^{-m}(\det \Sigma)^{-\frac{1}{2}n}(\det A)^{\frac{1}{2}(n-m-1)} \sum_{k=0}^{\infty} a_k \sum_{j=0}^{k} C_k^j [\operatorname{tr}(\Sigma^{-1}A+\Omega)]^{k-j}$$

$$\cdot [\operatorname{tr}(-2\Sigma^{-1}M'H_1T)]^j (dA)(H_1'dH_1) \quad . \tag{4}$$

here, we use

$$g(x) = \sum_{k=0}^{\infty} a_k x^k \qquad (-\infty < x < +\infty).$$

Because the integral on Stiefel manifold

$$\int_{H_1 \in V_{m,n}} [\operatorname{tr}(-2\Sigma^{-1}M'H_1T)]^j (H_1'dH_1)$$

$$= \frac{\Gamma_{n-m}[\frac{1}{2}(n-m)]}{2^{n-m}\pi^{\frac{1}{2}(n-m)^2}} \int_{H_1 \in V_{m,n}} \int_{K \in O(n-m)} [\operatorname{tr}(-2\Sigma^{-1}M'H_1T)]^j (K'dK)(H_1'dH_1)$$

$$= \frac{\Gamma_{n-m}[\frac{1}{2}(n-m)]}{2^{n-m}\pi^{\frac{1}{2}(n-m)^2}} \int_{H \in O(n)} [\operatorname{tr}(-2\Sigma^{-1}M'H_1T)]^j (H'dH)$$

$$= \frac{2^m \pi^{mn/2}}{\Gamma_m(\frac{1}{2}n)} \int_{O(n)} [\operatorname{tr}(-2\Sigma^{-1}M'H_1T)]^j (dH).$$

From Lemma 1, the last integral can be changed into

$$\int_{H_1 \in V_{m,n}} [\operatorname{tr}(-2\Sigma^{-1}M'H_1T)]^j (H_1'dH_1)$$

$$= \begin{cases} \dfrac{2^m \pi^{mn/2}}{\Gamma_m(\frac{1}{2}n)} 2^i (2i-1)!! \sum_{\kappa_1} \dfrac{C_{\kappa_1}(T\Sigma^{-1}M'M\Sigma^{-1}T')}{(\frac{1}{2}n)_{\kappa_1}} & j = 2i \\ 0 & j = 2i+1 \end{cases}$$

$$= \begin{cases} \dfrac{2^m \pi^{mn/2}}{\Gamma_m(\frac{1}{2}n)} 2^i (2i-1)!! \sum_{\kappa_1} \dfrac{C_{\kappa_1}(\Omega\Sigma^{-\frac{1}{2}}A\Sigma^{-\frac{1}{2}})}{(\frac{1}{2}n)_{\kappa_1}} & j = 2i \\ 0 & j = 2i+1 \end{cases} \tag{5}$$

where κ_1 is the same as κ in Lemma 1. Follow (4), A's density function is

$$\frac{\pi^{mn/2}}{\Gamma_m(\frac{1}{2}n)} (\det\Sigma)^{-\frac{1}{2}n} (\det A)^{(n-m-1)/2} \sum_{k=0}^{\infty} a_k \sum_{i=0}^{[\frac{k}{2}]} 2^i (2i)!! C_k^{2i} [\operatorname{tr}(\Sigma^{-1}A+\Omega)]^{k-2i} \cdot$$

$$\cdot \sum_{\kappa_1} \frac{C_{k_1}(\Omega\Sigma^{-\frac{1}{2}}A\Sigma^{-\frac{1}{2}})}{(\frac{1}{2}n)_{\kappa_1}} (dA) = \frac{\pi^{mn/2}}{\Gamma_m(\frac{1}{2}n)} (\det\Sigma)^{-\frac{1}{2}n} (\det A)^{(n-m-1)/2} \sum_{k=0} a_k$$

$$\sum_{i=0}^{[\frac{k}{2}]} \frac{k!}{(k-2i)!!} [\operatorname{tr}(\Sigma^{-1}A+\Omega)]^{k-2i} \sum_{\kappa_1} \frac{C_{\kappa_1}(\Omega\Sigma^{-\frac{1}{2}}A\Sigma^{-\frac{1}{2}})}{(\frac{1}{2}n)_{\kappa_1}} (dA) = \frac{\pi^{mn/2}}{\Gamma_m(\frac{1}{2}n)}$$

$$(\det\Sigma)^{-\frac{1}{2}n} (\det A)^{(n-m-1)/2} \sum_{\kappa=0}^{\infty} \frac{1}{k!} g^{(2k)} [\operatorname{tr}(\Sigma^{-1}A+\Omega)] \sum_{\kappa} \frac{C_\kappa(\Omega\Sigma^{-\frac{1}{2}}A\Sigma^{-\frac{1}{2}})}{(\frac{1}{2}n)_\kappa} (dA)$$

The proof is completeed.

Cacullos & Koutras[2], Fan[4] gave the density function of χ^2-distribution $G\chi_n^2(\delta^2; g)$ in integal form. But we can get it when $m=1$ in Theorem 1.

Corollary Assume g is Taylor expandiable on $(-\infty, +\infty)$, then we have that the density function of $G\chi_n^2(\delta^2; g)$ is

$$\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\delta^{2k}\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n+k)}u^{\frac{1}{2}n+k-1}g^{(2k)}(u+\delta^2), \quad u>0. \tag{6}$$

3. Latent Roots Distribution

We assume the latent root matrix of the generalized noncentral Wishart matrix A is $\Lambda=\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m)$, $\lambda_1>\lambda_2>\cdots>\lambda_m$, i.e., there exist orthogonal matrix $H\in O(m)$, such that $A=H\Lambda H'$.

Lemma 2. Assume $X_{n\times p}(n>p)$ is a random matrix, which is absolute continuous to Lebesgue measure in R^{np} space, B is a $n\times n$ symmetric matrix with rank r, $S=X'BX$, then the rank of S, $\rho(S)=\min(p, r)$, also the nonzero latent roots of S differ from each other in probability 1.

Lemma 3. Assume $Z\sim \mathrm{LEC}_{n\times m}(M, \Sigma, \varphi)$, $\Sigma>0$, $n>m$, and its density function exists, then $A=Z'Z$ is positive in probability 1.

Lemma 4. Assume $A=H\Lambda H'$ is a symmetric matrix with order m, $\Lambda=\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m)$, $\lambda_1>\lambda_2>\cdots>\lambda_m>0$ then

$$(\mathrm{d}A)=\frac{2^m\pi^{m^2/2}}{\Gamma_m(\frac{1}{2}m)}(\mathrm{d}H)\prod_{i<j}^{m}(\lambda_i-\lambda_j)\bigwedge_{i=1}^{m}\mathrm{d}\lambda_i \tag{7}$$

where $(\mathrm{d}H)$ is an invariate measure of $O(m)$.

The proof of Lemma 2 refer to [8]. The proof of Lemma 3 is similar to the method on P73 in [9]. The proof of Lemma 4 refer to P105 in [7].

Lemma 5[6] Assume X_1, X_2 are symmetrc matrices of order m, then we have

$$\int_{O(m)}C_\kappa(X_1HX_2H')(\mathrm{d}H)=\frac{C_\kappa(X_1)C_\kappa(X_2)}{C_\kappa(I_m)} \tag{8}$$

The latent roots distribution of noncentral Wishart matrix is given by James [5]. Chen [10] got the latent roots distribution of generalized central Wishart matrix. From all of the Lemmas, we can get the latent roots distribution of generalized noncentral Wishart matrix.

Theorem 2. Assume generalized noncentral Wishart matrix $A\sim \mathrm{GW}_m(n, \mathrm{I}, \Omega; g)$, and g is Taylor expansible on $(-\infty, +\infty)$. The latent roots matrix of A is $\Lambda=\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m)$, $\lambda_1>\lambda_2>\cdots>\lambda_m$, then the joint density function of $\lambda_1, \lambda_2, \cdots, \lambda_m$ is

$$\frac{2^{2m}\pi^{m(m+1)/2}}{\Gamma_m(\frac{1}{2}n)\Gamma_m(\frac{1}{2}m)}\prod_{i=1}^{m}\lambda_i^{(n-m-1)/2}\prod_{i<j}(\lambda_i-\lambda_j)\sum_{k=0}^{\infty}\frac{g^{(2k)}(\sum_{i=1}^{m}\lambda_i+\mathrm{tr}(\Omega))}{k!}\sum_{\kappa}\frac{C_\kappa(\Omega)C_\kappa(\Lambda)}{(\frac{1}{2}n)_\kappa C_\kappa(I_m)} \tag{9}$$

Proof From Lemma 2, 3, we have that $\lambda_1>\cdots>\lambda_m>0$ in probability 1. And from (7) we will have the joint density function of $\lambda_1,\lambda_2,\cdots,\lambda_m$.

$$f(\lambda_1,\lambda_2,\cdots,\lambda_m)=\frac{2^{2m}\pi^{m(m+1)/2}}{\Gamma_m(\frac{1}{2}n)\Gamma_m(\frac{1}{2}m)}\prod_{i=1}^{m}\lambda_i^{(n-m-1)/2}\prod_{i<j}(\lambda_i-\lambda_j)\ .$$
$$\cdot\sum_{k=0}^{\infty}\frac{g^{(2k)}[\sum_{i=1}^{m}\lambda_i+\mathrm{tr}\,\Omega]}{k!}\sum_{\kappa}\frac{1}{(\frac{1}{2}n)_\kappa}\int_{O(m)}C_\kappa(\Omega H\Lambda H')(dH)\ .$$

But from (8) we get the conclusion.

Corollary. Assume $A\sim GW_m(n,\Sigma,\Omega;g)$, and g is Taylor expansible on $(-\infty,+\infty)$, then the density function of the latent roots of matrix $\Sigma^{-\frac{1}{2}}A\Sigma^{-\frac{1}{2}}$ has the form in (9).

4. Generalized Variance Moment

Lemma 6 Assume Y is a symmetric matrix with of m, f is real function on $(0,+\infty)$, and

$$\sigma=\int_0^{\infty}f(z)z^{a}dz<\infty,\quad a=ma+k-1$$

$a>\frac{1}{2}(m-1)$, k is a given positive integer. If the left hand side integral of following formula exists, then we have

$$\int_{X>0}f(\mathrm{tr}X)(\det X)^{a-\frac{1}{2}(m+1)}C_\kappa(XY)(dX)=\frac{(a)_\kappa\Gamma_m(a)}{\Gamma(ma+\kappa)}\sigma C_\kappa(Y) \tag{10}$$

where κ is the partition of k, $(a)_\kappa$ refer to Lemma 1.

Proof Assume the left side of (10) is denoted by $h(Y)$, for every $H\in O(m)$, we have

$$h(HYH')=\int_{X>0}f(\mathrm{tr}X)(\det X)^{a-\frac{1}{2}(m+1)}C_\kappa(XHYH')(dX)$$

Now, we transform $U=H'XH$, then $(dU)=(dX)$, and

$$h(HYH')=h(Y)$$

i.e., h is a symmetric function of Y. Refer to Lemma 5, we have

$$h(Y)=\int_{O(m)}h(Y)(dH)=\int_{O(m)}h(HYH')(dH)$$
$$=\int_{X>0}f(\mathrm{tr}X)(\det X)^{a-\frac{1}{2}(m+1)}\int_{O(m)}C_\kappa(XHYH')(dH)(dX)$$
$$=\int_{X>0}f(\mathrm{tr}X)(\det X)^{a-\frac{1}{2}(m+1)}C_\kappa(X)C_\kappa(Y)/C_\kappa(I_m)(dX)$$

therefore, we get

$$h(Y)=h(I_m)C_\kappa(Y)/C_k(I_m). \tag{11}$$

Since $C_\kappa(Y)$ is a symmetric homogeneous polynomial in the latent roots of Y it can be assumed without loss of generality that y is diagonal, $Y=\mathrm{diag}(y_1, y_2, \cdots, y_m)$. Using the definition of $C_\kappa(Y)$ it follows that

$$h(Y)=\frac{h(\mathrm{I}_m)}{C_\kappa(\mathrm{I}_m)}\mathrm{d}_\kappa y_1^{k_1}\cdots y_m^{k_m}+\text{terms of lower weight,} \tag{12}$$

where $\kappa=(k_1,\cdots,k_m)$, On the other hand, using the resut of Lemma 7.2.6 in [7] we have

$$h(y)=\int_{X>0} f(\mathrm{tr}X)(\det X)^{a-\frac{1}{2}(m+1)}C_\kappa(XY)(\mathrm{d}X)$$

$$=\mathrm{d}_\kappa y_1^{k_1}\cdots y_m^{k_m}\int_{X>0} f(\mathrm{tr}X)(\det X)^{a-\frac{1}{2}(m+1)}\cdot$$

$$x_{11}^{k_1-k_2}\det\begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix}^{k_2-k_3}\cdots(\det X)^{k_m}(\mathrm{d}X)+\text{terms of lower weight,}$$

where $X=(x_{ij})_{m\times m}$. To evaluate this last integral, put $X=T'T$ where T is upper-triangular with positive diagonal elements. Then

$$\mathrm{tr}X=\sum_{i\le j}^{m}t_{ij}^2 \quad x_{11}=t_{11}^2, \quad \det\begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix}=t_{11}^2t_{22}^2,\cdots,\det X=\prod_{i=1}^{m}t_{ii}^2$$

and

$$(\mathrm{d}X)=2^m\prod_{i=1}^{m}t_{ii}^{m+1-i}\bigwedge_{i\le j}^{m}\mathrm{d}t_{ij},$$

so that, using Dirichlet integral we can obtain

$$h(Y)=\mathrm{d}_\kappa y_1^{k_1}\cdots y_m^{k_m}\int_{\substack{t_{ii}>0\\ -\infty<t_{ij}<\infty,\ i\ne j}}\cdots\int f\Big(\sum_{i\le j}^{m}t_{ij}^2\Big)\prod_{i=1}^{m}t_{ij}^{2a+2k_i-i}\cdot 2^m(\mathrm{d}T)+\cdots$$

$$=\mathrm{d}_\kappa y_1^{k_1}\cdots y_m^{k_m}\frac{\prod_{i=1}^{m}\Gamma(a+k_i-\frac{i-1}{2})\pi^{m(m-1)/4}}{\Gamma(\mathrm{m}a+k)}\int_0^\infty f(y)y^a\mathrm{d}y+\text{terms of lower weight.} \tag{13}$$

Equating coefficients of $y_1^k\cdots y_m^k$ in (12) and (13) it shows that

$$\frac{h(\mathrm{I}_m)}{C_\kappa(\mathrm{I}_m)}=\frac{(a)_\kappa\Gamma_m(a)}{\Gamma(ma+k)}\sigma$$

According to (11), we can prove Lemma 6.

Theorem 3. Assume $A\sim GW_m(n,\Sigma,\Omega;g)$, g is Taylor expansible on $(-\infty, +\infty)$, n or m is even, $r>0$ and

$$b(r)=\int_0^\infty g(y)y^{mr+\frac{1}{2}mn-1}\mathrm{d}y<\infty$$

If $E[(\det A)^r]$ exists, then we can get

$$E[(\det A)^{\gamma}] = \frac{\pi^{mn/2}\Gamma_m(r+\frac{n}{2})}{\Gamma_m(\frac{1}{2}n)\Gamma((\frac{mn}{2}+mr)}(\det\Sigma)^{r}$$

$$\cdot\{\sum_{k=0}^{a-1}\frac{1}{k!}\sum_{\kappa}\frac{(r+\frac{n}{2})_{\kappa}C_{\kappa}(\Omega)}{(\frac{n}{2})_{\kappa}}(a-1)\cdots(a-k)\int_{y>0}g(y+\mathrm{tr}\Omega)y^{a-k-1}\mathrm{d}y$$

$$+\sum_{k=a}^{\infty}\frac{1}{k!}\sum_{\kappa}\frac{(r+\frac{n}{2})_{\kappa}C_{\kappa}(\Omega)}{(\frac{1}{2}n)_{\kappa}}(a-1)!\,g^{(k-a)}(\mathrm{tr}\Omega)\} \quad (14)$$

where $a=m(\frac{1}{2}n+r)$.

Proof According to theorem 1, we get

$$E[(\det A)^{r}] = \frac{\pi^{mn/2}}{\Gamma_m(\frac{1}{2}n)}(\det\Sigma)^{\frac{n}{2}}\int_{A>0}(\det A)^{r+(n-m-1)/2}\sum_{k=0}^{\infty}\frac{g^{(2k)}[\mathrm{tr}(\Sigma^{-1}A+\Omega)]}{k!}\cdot$$

$$\cdot\sum_{\kappa}\frac{C_{\kappa}(\Omega\Sigma^{-\frac{1}{2}}A\Sigma^{-\frac{1}{2}})}{(\frac{n}{2})_{\kappa}}\mathrm{d}(A)$$

We transform $B=\Sigma^{-\frac{1}{2}}A\Sigma^{-\frac{1}{2}}$, and use the method of Lemma 6, then get that

$$E[(\det A)^{r}] = \frac{\pi^{mn/2}}{\Gamma_m(\frac{1}{2}n)}(\det\Sigma)^{r}\int_{B>0}(\det B)^{r+\frac{n-m-1}{2}}\sum_{k=0}^{\infty}\frac{g^{(2k)}[\mathrm{tr}B+\mathrm{tr}\Omega]}{k!}$$

$$\cdot\sum_{\kappa}\frac{C_{\kappa}(\Omega B)}{(\frac{1}{2}n)_{\kappa}}(\mathrm{d}B) = \frac{\pi^{mn/2}}{\Gamma_m(\frac{1}{2}n)}(\det\Sigma)^{r}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\kappa}\frac{C_{\kappa}(\Omega)}{(\frac{1}{2}n)_{\kappa}C_{\kappa}(I_m)}$$

$$\cdot\int_{B>0}(\det B)^{r+\frac{n-m-1}{2}}g^{(2k)}[\mathrm{tr}B+\mathrm{tr}\Omega]C_{\kappa}(B)(\mathrm{d}B) = \frac{\pi^{mn/2}}{\Gamma_m(\frac{1}{2}n)}(\det\Sigma)^{r}\Gamma_m(r+\frac{n}{2})\cdot$$

$$\cdot\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(a+k)}\sum_{\kappa}\frac{(r+\frac{n}{2})_{\kappa}C_{\kappa}(\Omega)}{(\frac{1}{2}n)_{\kappa}}\int_{y>0}g^{(2k)}[y+\mathrm{tr}\,\Omega]y^{a+k-1}\mathrm{d}y$$

$$=\frac{\pi^{mn/2}\Gamma_m(r+\frac{n}{2})}{\Gamma_m(\frac{1}{2}n)\Gamma(a)}(\det\Sigma)^{r}\cdot\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\kappa}\frac{(r+\frac{n}{2})_{\kappa}C_{\kappa}(\Omega)}{(\frac{1}{2}n)_{\kappa}}\int_{y>0}g^{(k)}[y+\mathrm{tr}\Omega]y^{a-1}\mathrm{d}y$$

The obtain of third equality sign in the last formula has used Lemma 6. There need some transforms then we get (13). The theorem is proved.

If $Z\sim LEC_{n\times m}(M,\Sigma,\varphi)$, it has a density function in form (1). But $Z\overset{d}{=}M+RU\Sigma^{1/2}$, where $R>0$ is a random variable. $\mathrm{Vec}\,U\overset{d}{=}u^{(nm)}$ is uniformly distributed on the unit sphere in R^{nm}, and independent on R. Then R's density function is

$$h(r)=\frac{2\pi^{\frac{1}{2}mn}}{\Gamma(\frac{1}{2}mn)}r^{mn-1}g(r^2).$$

Hence, in Theorem 3, if $\Omega=0$, $\int_{y>0} g(y)y^{a-k-1}\mathrm{d}y$ is just the moment form of R in certain order.

If we assume there exist all the variance moments of generalized noncentral Wishart matrix A, we can otain the formula of A's characteristic function in series form. Through some complex counting, we have following theorem.

Theorem 4. Assume $A\sim GW_m(n,\Sigma,\Omega;g)$, g is Taylor expansible on $(-\infty,+\infty)$, and for every $a>0$

$$\int_0^\infty g(y)y^a\mathrm{d}y<\infty$$

If it exists for A's generalized variance moments in any order, then A's characteristic function is,

$$\Phi(\Theta)=E[\exp(i\sum_{j\leq k}^{m}\theta_{jk}a_{jk})]$$

$$=\frac{2^m\pi^{mn/2}}{\Gamma_m(\frac{1}{2}n)}\sum_{k=0}^{\infty}\sum_{f=0}^{\infty}\frac{i^f}{k!f!}\sum_{\kappa}\sum_{\lambda}\sum_{\varphi\epsilon\kappa,\lambda}\frac{(\frac{n}{2})_\varphi C_\varphi^{\lambda,\kappa}(T,\Omega)C_\varphi^{\lambda,\kappa}(\mathrm{I},\mathrm{I})}{\Gamma(\frac{mn}{2}+f)(\frac{1}{2}n)_\kappa}$$

$$\cdot\int_0^\infty g^{(k)}(y+\mathrm{tr}\Omega)y^{nm/2+f-1}\mathrm{d}y$$

where $A=(a_{ij})_{m\times m}$, $\Gamma=(r_{ij})_{m\times m}$, $r_{ij}=(1+\delta_{ij})\theta_{ij}$, $\theta_{ij}=\theta_{ji}$, δ_{ij} is Kronecker delta function, $T=\frac{1}{2}\Sigma^{1/2}\Gamma\Sigma^{1/2}$, κ,λ are the partitions of k,f respectively. Other notes refer to [3].

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广义非中心Wishart分布

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摘　要

设Z是$n\times m$随机矩阵($n>m$)，它具有椭球等高分布$Z\sim \mathrm{LEC}_{n\times m}(M,\Sigma,\phi)$，$\Sigma>0$,其密度函数为

$$(\det\Sigma)^{-\frac{n}{2}}g[\mathrm{tr}((Z-M)\Sigma^{-1}(Z-M)')] \quad (1)$$

其中M是一个$n\times m$实矩阵，Σ是$m\times m$阶正定矩阵，g是一个适当的实函数．我们称$A=Z'Z$为广义非中心Wishart阵．它的分布函数记为$\mathrm{GW}_m(n,\Sigma,\Omega;g)$，此处$\Omega=\Sigma^{-\frac{1}{2}}M'M\Sigma^{-\frac{1}{2}}$为非中心参数．本文在一定的条件下，将给出级数形式的广义非中心Wishart分布的密度函数．特征根分布的密度函数、广义方差矩、特征函数的表达式．主要结果如下．

定理1　设g在$(-\infty,+\infty)$上可Taylor展开，则广义非中心Wishart阵A的分布密度可表为：

$$\frac{\pi^{mn/2}}{\Gamma_m(\frac{1}{2}n)}(\det\Sigma)^{-\frac{n}{2}}(\det A)^{(n-m-1)/2}\sum_{k=0}^{\infty}\frac{g^{(2k)}[\mathrm{tr}(\Sigma^{-1}A+\Omega)]}{k!}\sum_{\kappa}\frac{C_\kappa(\Omega\Sigma^{-\frac{1}{2}}A\Sigma^{-\frac{1}{2}})}{(\frac{1}{2}n)_\kappa} \quad (2)$$

$A>0$，其中κ为k的分划不超过m的分划．

定理2　设广义非中心Wishart阵$A\sim\mathrm{GW}_m(n,\mathrm{I},\Omega,g)$，且$g$可在$(-\infty,+\infty)$上Taylor展开．$A$的特征根阵为$\Lambda=\mathrm{diag}(\lambda_1,\lambda_2,\cdots,\lambda_m)$，$\lambda_1>\lambda_2>\cdots\ \lambda_m$．则$\lambda_1,\lambda_2,\cdots,\lambda_m$的联合分布密度为：

$$\frac{2^{2m}\pi^{m(m+1)/2}}{\Gamma_m(\frac{1}{2}n)\Gamma_m(\frac{1}{2}m)}\prod_{i=1}^{m}\lambda_i^{(n-m-1)/2}\prod_{i<j}(\lambda_i-\lambda_j)\cdot$$

$$\cdot\sum_{k=0}^{\infty}\frac{g^{(2k)}(\sum_{i=1}^{m}\lambda_i+\mathrm{tr}\Omega)}{k!}\sum_{\kappa}\frac{C_\kappa(\Omega)C_\kappa(\Lambda)}{(\frac{1}{2}n)_\kappa C_\kappa(\mathrm{I}_m)} \quad (3)$$

定理3　设$A\sim\mathrm{GW}_m(n,\Sigma,\Omega;g)$，$g$可在$(-\infty,+\infty)$上Taylor展开，$n$或$m$为偶数，$r>0$，且

$$b(r)=\int_0^{\infty}g(y)y^{mr+\frac{mn}{2}-1}\,dy<\infty.$$

如果 $E[(\det A)^r]$ 存在，则有

$$E[(\det A)^r]=\frac{\pi^{mn/2}\Gamma_m(r+\frac{n}{2})}{\Gamma_m(\frac{1}{2}n)\Gamma(\frac{mn}{2}+mr)}(\det\Sigma)^r.$$

$$\cdot\{\sum_{k=0}^{a-1}\frac{1}{k!}\sum_{\kappa}\frac{(r+\frac{n}{2})_\kappa C_\kappa(\Omega)}{(\frac{1}{2}n)_\kappa}(a-1)\cdots(a-k)\int_{y>0}g(y+\mathrm{tr}\Omega)y^{a-k-1}\mathrm{d}y$$

$$+\sum_{k=a}^{\infty}\frac{1}{k!}\sum_{\kappa}\frac{(r+\frac{n}{2})_\kappa C_\kappa(\Omega)(a-1)!}{(\frac{1}{2}n)_\kappa}g^{(k-a)}(\mathrm{tr}\Omega)\}. \tag{4}$$

其中 $a=m(\frac{n}{2}+r)$.

定理 4 设 $A\sim GW_m(n,\Sigma,\Omega;g)$，$g$ 在 $(-\infty,+\infty)$ 上可Taylor展开，又设对 $\forall a>0$，有

$$\int_0^\infty g(y)y^a\mathrm{d}y<\infty.$$

如果 A 的各阶广义方差矩都存在，则 A 的特征函数可表为

$$\Phi(\Theta)=E[\exp(i\sum_{j\leq k}^{m}\theta_{jk}Q_{jk})$$

$$=\frac{2^m\pi^{mn/2}}{\Gamma_m(\frac{1}{2}n)}\sum_{k=0}^{\infty}\sum_{f=0}^{\infty}\frac{i^f}{k!f!}\sum_{\kappa}\sum_{\lambda}\sum_{\phi\in\kappa,\lambda}\frac{(\frac{n}{2})_\varphi C_\varphi^{\lambda,\kappa}(T,\Omega)C_\varphi^{\lambda,\kappa}(\mathrm{I},\mathrm{I})}{\Gamma(\frac{mn}{2}+f)(\frac{n}{2})_\kappa}$$

$$\cdot\int_0^\infty g^{(k)}(y+\mathrm{tr}\Omega)y^{\frac{mn}{2}+f-1}\mathrm{d}y. \tag{5}$$

其中 $A=(a_{ij})_{m\times m}$，$\Gamma=(r_{ij})_{m\times m}$，$r_{ij}=(1+\delta_{ij})\theta_{ij}$，$\theta_{ij}=\theta_{ji}$，$\delta_{ij}$ 为Kronecker delta 函数，$T=\frac{1}{2}\Sigma^{\frac{1}{2}}\Gamma\Sigma^{\frac{1}{2}}$，$\kappa$、$\lambda$ 分别为 k, f 的分划.