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# Estimation of the Accuracy of Calculated Inverse Matrices*

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Many problems in practice (e.g.in the simplex method for solving LP) require computing an inverse of a nonsingular matrix. We always expect the calculated results to be as accurate as possible. How do yuo know the accuracy of the results? The estimate schemes now available give only an error on upper bounds that are usually amplified. This paper shows two useful theorems By means of them we can determine the accuracy of a calculate inverse matrix.

We consider the matrix equation

$$AX=B \tag{1}$$

for the unknow matrix $X$, where $A$ is a nonsingular $n\times n$ matrix, $B$ any $n\times n$ matrix.

To begin with, assume that all the elements of $A$ and $B$ are machine number (i.e., they can be exactly reprented in machine without any error).

**Lemma 1.** Let $X^*$ be the exact solution, $\widetilde{X}$ a numerical solution of system (1). If a nonsingular $n\times n$ matrix $U$ satisfies the inequality

$$\|I-UA\|<1, \tag{2}$$

then

$$\frac{\|UR\|}{1+\|I-UA\|}\leqslant\|\delta X\|\leqslant\frac{\|UR\|}{1-\|I-UA\|} \tag{3}$$

where $\delta X=\widetilde{X}-X^*$, $R=B-A\widetilde{X}$.

**Proof.** Because $R=B-A\widetilde{X}=AX^*-A\widetilde{X}=A\delta X$, $\delta X=-(UA)^{-1}UR=(I-I-UA)^{-1}UR$. Aoording to Banach lemma (see (1)), it follows that

$$\|(I-I-UA)^{-1}\|\leqslant 1/(1-\|I-UA\|).$$

Therefor

$$\|\delta X\|\leqslant\|UR\|/(1-\|I-UA\|). \tag{4}$$

From

$$\|UA\|=\|I-I-UA\|\leqslant 1+\|I-UA\|,$$

and

$$\|UR\|=\|UA\delta X\|\leqslant\|UA\|\,\|\delta X\|$$

we immediately obtain

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$$\|\delta X\| \geqslant \|UR\| / \|UA\| \geqslant \|UR\| / (1 + \|I - UA\|) \tag{5}$$

combining (4) with (5), the proof is complete.

It should be pointed that the inequality (3) holds also when $B$ is an $n \times n$ matrix in system (1), provided that we use consistent matrix norms.

In particular, for $B = I$, an numerical solution $\widetilde{X}$ of system (1) is also an approximation of $A^{-1}$. If inequality (2) holds for $U = \widetilde{X}$, we can take it as $U$. From this, we give the follow theorem.

**Theorem 1** Let $n \times n$ matrix $A$ be nonsingular and let $\widetilde{X}$ be a numerical solution of matrix equation $AX = I$ (i.e., $\widetilde{X}$ is an approximate inverse of $A^{-1}$). If the matrix $\widetilde{X}$ is nonsingular and satisfies $\|I - XA\| < 1$, then

$$\frac{\|\widetilde{X}R\|}{1 + \|F\|} \leqslant \|\widetilde{X} - A^{-1}\| \leqslant \frac{\|\widetilde{X}R\|}{1 - \|F\|} \tag{6}$$

$$\frac{\|\widetilde{X}R\|}{(1 + \|F\|)\|\widetilde{X}\|} \leqslant \frac{\|\widetilde{X} - A^{-1}\|}{\|\widetilde{X}\|} \leqslant \frac{\|\widetilde{X}R\|}{(1 - \|F\|)\|\widetilde{X}\|} \tag{7}$$

where $R = I - A\widetilde{X}$, $F = I - \widetilde{X}A$.

From (6), (7) it is seen that the error bounds of the matrix $\widetilde{X}$ with respect to $A^{-1}$ would be amplified 10 at most when $F \leqslant 0.9$, and $1/0.9$ at most when $F \leqslant 0.1$. Thus estimate (7) can indicate the number of significant digits of the elements of the matrix $\widetilde{X}$ in respect of $A^{-1}$.

Now assume that the elements of $A$ or $B$ are not machine numbers. We have the following Lemma.

**Lemma 2** Let $\overline{A}$ and $\overline{B}$ be input data of $A$ and $B$ respectively, and let $\overline{A}$ be nonsingular. Suppose $X^*$ is the exact solution of matrix equation $AX = B$, $\widetilde{X}$ an numerical solution of $\overline{A}X = \overline{B}$. Let $R = \overline{B} - \overline{A}X$, $F = I - X\overline{A}$, and let $\delta X = \widetilde{X} - X^*$, such that $\|\delta X\| \leqslant r\|X^*\|$ With $0 \leqslant r \leqslant 1$. If a nonsingular $n \times n$ matrix $U$ satisfies $\|UA - I\| < 1$ then

$$\frac{\|UR\|}{1 + \|F\|} \leqslant \|\delta X\| \leqslant \frac{\eta\|U\|[(1 + r)\|\overline{A}\|\|\widetilde{X}\| + \|\overline{B}\|] + \|UR\|}{1 - \|F\|} \tag{8}$$

here $\eta$ is machine precision.

**Proof.** We show at only the right side of (8). Let $\overline{X}^*$ be the eact solution of the equation $\overline{A}\overline{X} = \overline{B}$. Let $\delta A = A - \overline{A}$, $\delta B = B - \overline{B}$, and satisfy $\|\delta A\| \leqslant \eta\|\overline{A}\|$, $\|\delta B\| \leqslant \eta\|\overline{B}\|$. in which $\delta_R X = \widetilde{X} - \overline{X}^*$, $\delta_D X = \overline{X}^* - X^*$. Therefor

$$\|\delta X\| \leqslant \|\delta_D X\| + \|\delta_R X\| \tag{9}$$

On the other hand,

$$\overline{A}X^* - \overline{A}\overline{X}^* = \delta B - \delta AX^* = -\overline{A}\delta_D X,$$

$$U\overline{A}\delta_D X = U(\delta AX^* - \delta B)$$

$$\delta_D X = (U\overline{A})^{-1}U(\delta AX^* - \delta B)$$

$$\|\delta_D X\| \leqslant \eta n\|(U\overline{A})^{-1}\|\|U\|(\|\overline{A}\|\|X^*\| + \|\overline{B}\|) \tag{10}$$

Similarly. We have

$$\|\delta_R X\| < \|(U\bar{A})^{-1}\| \|UR\| \tag{11}$$

From ( 9 )—(11), we obtain ( 8 ) by using Banach lemma.

Note That estimate ( 8 ) is suitable for rectangular matrix under consistant matrix norms. Especially, in the case of $B = I$, $\delta B$ vanishes. The following theorem is analogous to theorem 1.

**Theorem 2** Let input matrix $\bar{A}$ of matrix $A$ be nonsingular and let $\widetilde{X}$ be approximate inverrse of $A$.If $\widetilde{X}$ is nonsingular and satisfies the conditions:

1. $\|\widetilde{X} - A^{-1}\| < r\|\widetilde{X}\|$ with $0 < r < 1$
2. $\|I - \widetilde{X}\bar{A}\| < 1$.

then

$$\frac{\|\widetilde{X}R\|}{1 + \|F\|} < \|\widetilde{X} - A^{-1}\| < \frac{\eta(1+r)\|\widetilde{X}\|^2\|\bar{A}\| + \|\widetilde{X}R\|}{1 - \|F\|} \tag{12}$$

$$\frac{\|\widetilde{X}R\|}{(1 + \|F\|)\|\widetilde{X}\|} < \frac{\|\widetilde{X} - A^{-1}\|}{\|\widetilde{X}\|} < \frac{\eta(1+r)\|\widetilde{X}\|^2\|\bar{A}\| + \|\widetilde{X}R\|}{(1 - \|F\|)\|\widetilde{X}\|} \tag{13}$$

where $R = I - \bar{A}\widetilde{X}$, $F = I - \widetilde{X}\bar{A}$, $\eta$ is the machine precision.

In general, the solutions of a matrix equation are affected by roundoff erroes much more than by input errors. So in examining the accuracy of calculate results, the inequality ( 3 ), ( 6 ) and ( 7 ) can replace ( 8 ), (12) and (13), respectively.

Example. Given

$$A = \begin{bmatrix} 0.20000000 & 0.40000000 \\ 0.20000000 & 0.40000010 \end{bmatrix}$$

we find easily

$$A^{-1} = \begin{bmatrix} 0.20000005 & -0.20000000 \\ -0.10000000 & 0.10000000 \end{bmatrix} \times 10^8$$

if

$$\widetilde{X} = \begin{bmatrix} 0.20000006 & -0.20000000 \\ -0.10000000 & 0.10000000 \end{bmatrix}$$

then

$$\|\widetilde{X} - A^{-1}\|_\infty / \|A^{-1}\|_\infty \doteq 0.25 \times 10^{-7}$$

Thus $X$ should have seven number of significant digits in respectt of $A^{-1}$. However, according to the estimate used usually

$$\frac{\|\widetilde{X} - A^{-1}\|}{\|A^{-1}\|} < \|I - A\widetilde{X}\|,$$

we obtain

$$\frac{\|\widetilde{X} - A^{-1}\|_\infty}{\|A^{-1}\|_\infty} < 0.2$$

from

$$I - A\widetilde{X} = \begin{bmatrix} -0.2 & 0 \\ -0.2 & 0 \end{bmatrix}$$

This implies $X$ could not have any significant digit.

Now we use inequality (7). Since $F = \begin{bmatrix} -0.2 & 0.4 \\ 0 & 0 \end{bmatrix}$ and $XR = \begin{bmatrix} -1.2 & 0 \\ 0 & 0 \end{bmatrix}$.

$$0.18 \times 10^{-7} < \frac{\|\widetilde{X} - A^{-1}\|}{\|\widetilde{X}\|} < 0.75 \times 10^{-7}$$

This determines precisely that $X$ has seven significant digits.

## References


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# 矩阵计算逆的精度估计


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在许多问题中，需要计算非奇异矩阵的逆矩阵．然而算得的逆矩阵相对于精确的逆矩阵来说，究竟有几位有效数字，往往是不得而知的．本文给出的定理 1 和定理 2，能在算得的逆矩阵与原矩阵之间满足了一个要求不算高的关系式之后，准确地判断出算得的逆矩阵有几位有效数字．