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Projectivity and Injectivity of Simple Modules over Commutative Rings*

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In this short paper we investigate the projectivity and injectivity of simple modules over commutative rings.

Throughout this paper, R is an associative commutative ring with identity and all modules are unital modules.

Theorem 1. If P_R is a projective simple module, Then P_R is an injective module.

Proof. Since P_R is a projective simple module. Hence there is a maximal ideal M and a minimal ideal I of R with $R\cong M\oplus I$ where $I_R\cong P_R$. Let e_1 and e_2 be projection of 1 in M and I respectively. Then it is easily to verify that e_1 and e_2 are two orthogonal idempotents of R and $M=e_1R$ and $I=e_2R$. To establish our result we only need to prove that I is an injective module. Let J be an arbitrary ideal of R and $f:J\to I$ be a homomorphism. (i) If $J\cap I=0$. Then $f(J)=f(J)e_2=f(Je_2)$. But $Je_2\subseteq J\cap I=0$. Hence $f(J)=0$, that is, f can extend to R. (ii) Or $J\cap I\neq 0$. Since I is a simple module. Hence $I\subseteq J$. And then $R=J+M$. Hence $1=y+m$ where $y\in J$ and $m\in M$. We define an R-homomorphism $\hat{f}:R\to I$ via $\hat{f}(r)=f(ye_2)r$. When $r\in J$, $\hat{f}(r)=f(ye_2)r=f(ye_2r)=f((1-m)e_2r)=f(e_2r)-f(me_2r)$. But $e_2m\in M\cap I=0$. Hence $f(me_2r)=0$. Thus $\hat{f}(r)=f(r)$. That is, f can extend to R. Hence I_R is injective.

Thorem 2 Let R be a commutative Noetherian ring and M_R be an injective module. Then M_R is projective.

Before we prove our theorem, let us do some works for preparation.

Lemma 1 Let M be a simple module. Then for any $P\in\mathrm{Spec}(R)$, either M_P is a simple R_P-module or $M_P=0$.

Proof If $M_P\neq 0$ and N is a nonzero submodble of M_P as R_P-module. And then N also is a nonzero submodule of M_P as R-module. Since M is an essential submodule of M_P as R-module. Hence $N\cap M\neq 0$. Then $M\subseteq N$ as R-module by simpicity

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of M. Then we have an exact sequence $0 \to M_P \to N_P$ since the functor of localization is an exact functor. Since N is an R_P-module. Hence $N_P = N$ by [3, Lemma 3.75]. Thus we have $N = M_P$, i.e., M_P is a simple module.

Lemma 2 Let $f: K_R \to M_R$ and $g: M_R \to N_R$ be two R-homomorphisms. Then the following conditions are equivalent.

(i) $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ is an exact sequence.

(ii) For any $P \in \mathrm{Spec}(R)$, $0 \to K_P \xrightarrow{f} M_P \xrightarrow{g} N_P \to 0$ is an exact sequence.

Proof (i)$\Rightarrow$(ii) We refer to [2, Prop. 3.3].

(ii)$\Rightarrow$(i). By [2, Prop. 3.9] we know that f is a momorphism. And $Im(gf) = 0$ by [2, Prop. 3.8]. If g is not an epimorphism, Then there is an exact sequnce $M \to N \to N/Im(g) \to 0$ and $N/Im(g) \neq 0$. By [2, Prop. 3.8] we know that there is a $P \in \mathrm{Spec}(R)$ such that $(N/Im(g))_P \neq 0$. But this contradict to condition (ii). Hence g is an epimorphism. If $Im(f) \subsetneq \mathrm{Ker}(g)$. Then there is some $P \in \mathrm{Spec}(R)$ such that $Im(f)_P \subsetneq \mathrm{Ker}(g)_P$ and this also contradict to the condition (ii) Hence the sequence in (i) is exact.

Lemma 3 M_R is a finitely presented R-module. Then there is anatural isomorphism $(\mathrm{Hom}_R(M, N))_P \cong \mathrm{Hom}_{R_P}(M_P, N_P)$ for any $P \in \mathrm{Spec}(R)$ and any R module N.

Proof Since M_R is finitely presented. Hence there are two natural numbers mandn such that $R^{(m)} \to R^{(n)} \to M \to 0$. Since $\mathrm{Hom}_R(M, N)$ is a left exact contravariant functor. Hence we have following exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \to N^{(n)} \to N^{(m)}.$$

Similar as above we also have

$$0 \to \mathrm{Hom}_{R_P}(M_P, N_P) \to N_P^{(n)} \to N_P^{(m)}.$$

And then we have following commutative diagram

$$\begin{array}{ccccc} 0 \to (\mathrm{Hom}_R(M, N))_P & \to & N_P^{(n)} & \to & N_P^{(m)} \\ \downarrow \mu_{M_P} & & \wr\!\wr & & \wr\!\wr \\ 0 \to \mathrm{Hom}_{R_P}(M_P, N_P) & \to & N_P^{(n)} & \to & N_P^{(m)} \end{array} .$$

From the commutative diagram we know that μ_{M_P} is a momorphism. And by [1, Lemma 3.14] we know that μ_{M_P} is an isomorphism. Thus we have proved the lemma.

Now we are in the position to prove the theorem 2.

Proof of theorem 2 For any $P \in \mathrm{Spec}(R)$ and any ideal J of R_P, there is an ideal I of R with $J = I_P$ by [2, Prop. 3.11]. Clearly I satifies the condition of lemma 3. Thus we a commutative diagram with isomorphic columns

$$\begin{array}{ccccccc} 0 \to \mathrm{Hom}_{R_P}(R_P/J, M_P) & \to & M_P & \to & \mathrm{Hom}_{R_P}(J, M_P) & \to & 0 \\ \wr\!\wr & & \wr\!\wr & & \wr\!\wr & & \\ 0 \to (\mathrm{Hom}_R(R/I, M))_P & \to & M_P & \to & (\mathrm{Hom}_R(I, M))_P & \to & 0 \end{array}$$

Since M_R is injective and the functor of localization is exact, we obtain an exact sequence

$$0 \to \mathrm{Hom}_{R_P}(R_P/J, M_P) \to M_P \to \mathrm{Hom}_{R_P}(J, M_P) \to 0,$$

that is, M_P is either an injective simple module or a zero module by emma 1. Since R_P is a local ring, the irredundant set of representatives of the simple modules in Mod-R_P has only one element. Hence M_P is either a cogenerator or a zero module by [1,Cor. 18.16]. If $M_P \neq 0$, then R_P has a faithful simple module M_P, and then R_P is a commutative primitive ring. Hence R_P is a field. Then it is clearly that M_P is a projective R_P-module. If $M_P = 0$, M_P is also a projective R_P-module. Thus for any $P \in \mathrm{Spec}(R)$ and any short exact sequence

$$0 \to K_R \to N_R \to L_R \to 0,$$

we have an short exact sequence

$$0 \to \mathrm{Hom}_{R_P}(M_P, K_P) \to \mathrm{Hom}_{R_P}(M_P, N_P) \to \mathrm{Hom}_{R_P}(M_P, K_P) \to 0.$$

It is clearly that M_R satisfies the condition of lemm 3 since M_R is a simple module and R is a Noetherian ring. Thus we obtain an exact sequence as follows

$$0 \to (\mathrm{Hom}_R(M, K))_P \to (\mathrm{Hom}_R(M, N))_P \to (\mathrm{Hom}_R(M, L))_P \to 0.$$

The arbitrariness of P implies that

$$0 \to \mathrm{Hom}_R(M, K) \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, L) \to 0.$$

And the arbitrariness of exact sequence implies that M_R is a projective module.

One may ask that if the above results hold for noncommutative ring. Here we shall give an example to show that the results will not hold for non-commutative ring even if it is two side Artinian.

Example We denote Q as field of rational number. And $R = \begin{pmatrix} Q & Q \\ 0 & Q \end{pmatrix}$. It is easily to verify that R is a two side Artinian ring and $I = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}$ is a simple projective right ideal of R. Since for any $r \in R$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} r = 0$ if and only if $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} r = 0$. Thus we can construct an R-homomorphism

$$f: \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix} \to I \quad \text{via} \quad f\left(\begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}$$

which can not extend to R. For otherwise, there must be an element $\begin{pmatrix} 0 & 0 \\ 0 & q_1 \end{pmatrix}$ such that $f\left(\begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & q_1 \end{pmatrix}\begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix}$ by Baer's criterion. But this equation can not be hold. Hence I is not an injective module.

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交换环上的单模的射影性与内射性

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摘　要

在本短文中主要研究了交换环上的单模的射影性与内射性之间的关系。主要证明了：交换环上射影单模必是一个内射模；若该环是一个Noether环，则反之亦成立.

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关于周期环结构的几个定理

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摘　要

本文给出了有关周期环结构的一些结果，而使文献[1]中的所有定理都可以做为本文结果的直接结论.