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# Some Theorems Concerning Structure of Periodic Rings*

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A ring $R$ is called a periodic ring, if for every $x\in R$ there is a pair $n(x)$, $m(x)$ of distinct positive integers such that $x^{n(x)}=x^{m(x)}$. Abu-Khuzam and Yaqub [1], Bell [2], and Xie Bangjie [3] studied the properties of periodic rings and obtained some structure theorems concerned.

The purpose of this paper is to generalize the theorems of Abu-Khzam and Yaqub [1].

Throughout this paper $R$ is a associative ring, $C$ is the central of $R$, $N$ and $D$ denote the set of nilpotents and the set of right zero divisors of $R$, respectively.

**Theorem I.** In $R$, if the conditions:

(i) for each $a,b\in N$, there exists a integer $n(a,b)>1$ for which $(ab-ba)^{n(a,b)}=ab-ba$;

(ii) every $x$ in $R$ can be uniquely written in the form $x=e+a$, where $e^2=e$ and $a\in N$,

are satisfied, then $R$ is commutative, and $R/N$ is Boolean.

**Proof.** There are two cases.

*Case* 1. $R=N$. A wel l known theorem of Herstein ([4], Theorem 6) implies that $R$ is commutative. Clearly, $R/N$ is Boolean.

*Case* 2. $R\neq N$. Frist, it follows from (ii) that al l idempotents of $R$ are central (see [1], the proof of Theorem 1). Let $x,y\in R$ and $x=e+a$, $y=f+b$, where $e^2=e$, $f^2=f$ and $a,b\in N$. Since $(ab-ba)^{n(a,b)}=ab-ba$ and $xy-yx=ab-ba$, we have

$$(xy-yx)^{n(a,b)}=(ab-ba)^{n(a,b)}=ab-ba=xy-yx. \qquad (1)$$

A well known theorem of Herstein [4] asserts that (1) implies that $R$ is commutative and hence $N$ forms an ideal of $R$. We know that $R/N$ is Boolean by (ii).

As an immediate consequence, we have the following

**Corollary I** ([1], Theorem 1) Let $R$ be a periodic ring. Suppose that (i) $N$ is commutative, and (ii) every $x$ in $R$ can be uniquely written in the form $x=e+a$, where $e^2=e$ and $a\in N$. Then $N$ is an ideal in $R$, and $R/N$ is Boolean. In

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fact, $R$ is commutative.

**Theorem 2.** In $R$, if the conditions:

(A) for each $a, b \in N$, there exists $c \in C$ ($c$ depends on $a$ and $b$) such that $ba = cab$;

(ii) the same as (ii) of Theorem 1,

are satisfied, then $N$ is an ideal of $R$, and $R/N$ is Boolean.

**Proof.** We have only to show that $N$ is an ideal of $R$. Since the condition (A) is satisfied, hence $N$ is a subring of $R$. It follows from (ii) that all idempotents of $R$ are central. Hence $xb = (e+a)b = eb + ab \in N$, where $a, b \in N$, $e^2 = e$ and $x = e + a$. Thus $N$ is an ideal of $R$.

An ideal $P$ is called reflexive if $xy \in P$ implies $yx \in P$. Clearly, every reflexive prime ideal is completely prime. An element $x$ in $R$ is right semi-central if $xr = xrx$ for each $r \in R$.

We are now in a position to prove the following that improves Theorem 2 of [1].

**Theorem 3.** If a periodic ring $R$ satisfies the following:

(i) for any $a, b \in N$ there exists $r \in R$ ($r$ depends on a and $b$) such that $ba = rab$;

(ii) every $x$ in $D$ can uniquely written in the form $x = e + a$, where $e^2 = e$ and $a \in N$,

then $N$ coincides with the Bare radical of $R$. In particular, if $R \neq N$, then $R/N$ is isomorphic to a subdirect sum of fields.

**Proof.** We prove first that all idempotents of $R$ are right semi-central. Let $e^2 = e \in R$, $x \in R$ and let

$$f = e + (ex - exe). \tag{2}$$

Then $f^2 = f$, $(ex - exe)^2 = 0$, and $(ex - exe)f = 0$. If $ex - exe \neq 0$, then $f \in D$. Also since (2) and

$$f = f + 0.$$

it follows from (ii) that $ex - exe = 0$, i.e., $ex = exe$.

Next we assert that every prime ideal of $R$ is reflexive. Let $P$ be any prime ideal in $R$ and suppose $xy \in P$. If $y \notin D$, then there exists a integer $k(y) > 1$ such that $y^{k(y)} = y$. Thus $y^{k(y)-1}$ is idempotent, and hence $y^{k(y)-1}x = y^{k(y)-1}xy^{k(y)-1} \in P$. So $yx \in P$. Similarly, if $x \notin D$, then $x^{k(x)-1}Rx^{k(x)-1}y = x^{k(x)-1}Ry \subseteq P$, where $x^{k(x)-1}$ is idempotent. Thus $yx^{k(x)-1} \in P$, and hence $yx \in P$. Now, let $x, y \in D$ and $x = e + a$, $y = f + b$, where $e^2 = e$, $f^2 = f$ and $a^n = 0, b^m = 0$. Since

$$xy = ef + eb + af + ab, \tag{3}$$

then

$$(e - ea + \cdots + (-e)^{n-1}a^{n-1})exy = (e - ea + \cdots + (-e)^{n-1}a^{n-1})(e + ea)(ef + eb) = ef + eb \in P \tag{4}$$

Also, $fe=fef=(fe)^2$. Call $g=fe$, thus

$$f(ef+eb)(g-gb+\cdots+(-g)^{m-1}b^{m-1})$$
$$=g(g+gb)(g-gb+\cdots+(-g)^{m-1}b^{m-1})=g^2=fe\in P, \quad (5)$$

and

$$ef=efe=egf\in P \quad (6)$$

Using (4) and (6) we get

$$eb\in P \quad (7)$$

Thus $eRb=eReb\subseteq P$. Hence

$$be\in P \quad (8)$$

Moreover, by (3) and (4) we have

$$af+ab\in P \quad (9)$$

therefore $f(af+ab)\in P$. Similarly we also have

$$fa\in P \text{ and } af\in P \quad (10)$$

Using (3), (4) and (10) we get

$$ab\in P \quad (11)$$

Since

$$yx=fe+be+fa+ba=fe+be+fa+rab \quad (12)$$

where $ba=rab$, combining (12) with (5), (8), (10) and (11), we see $yx\in P$. Hence every prime ideal of $R$ is reflexive. Therefore $N$ coincides with the Bare radical of $R$.

Now, since $\overline{R}=R/N$ is periodic and Bare-semisimple, we get that $\overline{R}$ is isomorpnic to a subdirect sum of prime rings. As prime ideal of $\overline{R}$ is also reflexive, then it is completely prime. Also, a periodic ring without zero divisor is a field ([3], Theorem2). Therefore $\overline{R}$ is isomorphic to a subdirect sum of fields.

We can easily obtain ([1], Theorem 2) as a corollary of Theorem 3.

**Corollary 2.** Let $R$ be a periodic ring with identity 1. Suppose that (i) $N$ is commutative, (ii) every $x\in D$ can be uniquely written in the form $x=e+a$, where $e^2=e$ and $a\in N$. Then $N$ is an ideal in $R$ and $R/N$ is isomorphic to a subdirect sum of fields.

**Theorem 4.** If a periodic ring $R$ with identity 1 satisfies the following:

(i) the same as (A) of Theorem 2.

(ii) every $x$ in $D$ is either idempotent or nilpotent, then $N$ is an ideal of $R$, and $R/N$ is either Boolean or a field.

**Proof.** We prove first $N$ is an ideal of $R$. Since every $x$ in $R$ is nilpotent or idempotent or a unit, and $N$ is subring, there are two cases.

*Case* 1. Let $a\in N$ and $x^2=x\in R$. Using $ax-xax\in N$ and (A) we get $a(ax-xax)=c(ax-xax)a$, where $c\in C$. Now multipling the above equation from the left and right by $x$ respectively, we get $xa^2x-xaxax=c(xaxax-xaxax)=0$, i.e., $(xax)^2=$

$xa^2x$. Using $a^2$ in place of $a$ in the above equation, we have $(xax)^4=(xa^2x)^2=xa^4x$. By induction, we get $xax\in N$. Thus $ax\in N$, and hence $xa\in N$.

*Case* 2. Let $a^n=0, a^{n-1}\neq 0$ $(n>1)$ and $1\neq x\notin D$. We prove that $ax\in N$. First of all, we assert that $ax\in D$. If $ax\notin D$, there exists a integer $k=k(ax)>1$ such that $(ax)^k=1$. Multipling the above equation by $a^{n-1}$ from the left, we get $a^{n-1}=0$, which is a contradiction to $a^{n-1}\neq 0$. Hence $ax\in D$.

Suppose $(ax)^2=ax$. Then $a(xa-1)=0$, and hence $xa-1\in D$. Therefore either $xa-1\in N$ or $(xa-1)^2=xa-1$. If $xa-1\in N$, there exists $c\in C$ such that $(xa-1)a=ca(xa-1)=0$. Hence $a=xa^2=x^{n-1}a^n=0$. This is a contradiction to $a\neq 0$. Also, if $(xa-1)^2=xa-1$. We have $(xa-1)a\in N$ by case 1. Thus there exists $c\in C$ such that $(xa-1)a^2=ca(xa-1)a=0$, i.e., $a^2=xa^3=x^{n-2}a^n=0$. Since $1\neq x\notin D$, we can suppose that $x^m=1$ and integer $m>1$. It follows that $a-x^{m-1}\in D$ from $0=a(xa-1)=ax(a-x^{m-1})$. Since $a$ is in the subring $N$, $a-x^{m-1}$ must be idemptent. From Case 1 and the fact which $a-x^{m-1}$ is idempotent, we get $a(a-x^{m-1})\in N$. So $ax^{m-1}\in N$ and $x^{m-1}a\in N$. Thus there exists $c\in C$ such that

$$ax^{m-2}a=(ax^{m-1})(x^{m-1}a)=c(x^{m-1}a)(ax^{m-1})=0.$$

Hence $ax^{m-2}\in N$. Similarly, $ax^2\in N$. Then

$$a=axa=(ax^2)(x^{m-1}a)=c(x^{m-1}a)(ax^2)=0.$$

This is a contradiction to $a\neq 0$. Thus $ax\in N$, and hence $xa\in N$. Therefore $N$ is an ideal of $R$.

Let $\overline{R}=R/N$. If $\overline{R}$ has no zero divisors, it follows that $\overline{R}$ is a field from ([3], Theorem 2).

If $\overline{R}$ has zero divisors, then all zero divisor of $\overline{R}$ are idempotent. For if $\overline{x}\overline{y}=0$ and $\overline{x}\neq 0, \overline{y}\neq 0$, then $xy\in N$ and $x\notin N$, $y\notin N$, Hence $x, y$ are idempotent, so are $\overline{x}, \overline{y}$. Also, since $\overline{R}$ is periodic without nonzero nilpotent elements, we get that $\overline{R}$ is a J-ring (i.e., one with Jacobson's $x^{n(x)}=x$ property) from Theorem 1 of [3]. Thus $\overline{R}$ is a commutative ring. For any $\overline{x}\notin D$, we take a nonzero idempotent $\overline{e}$ which is not identity $\overline{1}$ out of $\overline{R}$. Since $\overline{e}(\overline{e}-\overline{1})=0$. then $(\overline{e}-\overline{1})^2=\overline{e}-\overline{1}$. Also, it follows that $(\overline{e}\overline{x}-\overline{x})^2=\overline{e}\overline{x}-\overline{x}$ from $\overline{e}(\overline{e}\overline{x}-\overline{x})=0$. Hence $(\overline{e}-\overline{1})(\overline{1}-\overline{x})=0$, and so $(\overline{1}-\overline{x})^2=\overline{1}-\overline{x}$. Thus $\overline{x}^2=\overline{x}$. This proves that $\overline{R}$ is a Boolean ring

As an immediate consequence of the above theorem, we have the following.

**Corollary 3** ([1], Theorem 3) Let $R$ be a periodic ring with identity 1. Suppose that (i) $N$ is commutative, and (ii) every $x$ in $D$ is either idempotent or nilpotent, Then $N$ is an ideal of $R$ and $R/N$ is either Boolean or a field.

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# 交换环上的单模的射影性与内射性

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## 摘　要

在本短文中主要研究了交换环上的单模的射影性与内射性之间的关系。主要证明了：交换环上射影单模必是一个内射模；若该环是一个Noether环，则反之亦成立.

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## References

[1] Abu-Khuzam, Hazar and Yaqub, Adil, Canad. Math. Bull., 28(1985), no.1, 120-123.

[2] Bell. H. E., Acta Math. Acad. Sci. Hungar, 28(1976), no.3-4, 279-283.

[3] Xie Bangjie, J. Math. Res. and Exposition, 2(1982), no.2, 11—13.

[4] Herstein. I. N., Canad. J. Math., 9(1957), 583—586.

# 关于周期环结构的几个定理

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## 摘　要

本文给出了有关周期环结构的一些结果，而使文献[1]中的所有定理都可以做为本文结果的直接结论.