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On Generalized A-Groups*

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P.Hall defined the concept of A-groups in [1] as follows.

Definition 1. A finite solvable group is an A-group if all of its Sylow subgroups are abelian.

P.Hall mentioned the following theorems without proofs.

Theorem A. Let G be an A-group. Then $G'\cap Z(G)=1$.

Theorem B. Let G be an A-group and let N be the system normalizer of G. Then $G=G'N$ and $G'\cap N=1$.

D.R.Taunt proved these theorems in [2]. D.R.Taunt, B.Huppert and R.W.Carter studied the structure theory of A-groups (See [2], [3] and [4]).

In this paper we define the concept of so-called "generalized A-group", which are weaker than that of A-groups, and generalize the properties of A-groups to the generalized A-groups.

Definition 2. A finite solvable group G is called GA-group, i.e., generalized A-group if derived group of every Sylow subgroup of G is contained in the centre of G.

Obviously, an A-group is a GA-group but the converse is not true.

Throughout this paper, let $p_1, p_2, \cdots, p_r$ be all of the distinct prime divisors of order of G and let $P_1, P_2, \cdots, P_r$ be Sylow subgroups corresponding to the prime divisors. Suppose that G is a GA-group, by definition 2, then $P_1'P_2'\cdots P_r'=P_1'\times P_2'\times\cdots\times P_r'\leqslant Z(G)$.

Proposition 1. Let G be a GA-group and $P_1'\times P_2'\times\cdots\times P_r'\leqslant H\trianglelefteq G$, then G/H is an A-group. Conversely, let G/H be an A-group and $H\leqslant Z(G)$, then G is a GA-group and $P_1'\times P_2'\times\cdots\times P_r'\leqslant H$. Thus, in fact, a GA-group is a central extension of an abelian group by an A-group.

Proof. Suppose that G is a GA-group and $P_1'\times P_2'\times\cdots\times P_r'\leqslant H\trianglelefteq G$. Then the Sylow subgroups of G/H are $P_iH/H (i=1,2,\cdots,r)$. Since $[xH, yH]=[x,y]H\in P_i'H=H$ for x, y in P_i, P_iH/H is abelian. So G/H is an A-group. Conversely, suppose that G/H is an A-group and $H\leqslant Z(G)$. Then for a Sylow subgroup P of G, by

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G/H being an A group and PH/H being a Sylow subgroup of G/H, PH/H is abelian. Thus, $[P,P]H/H=[PH/H, PH/H]=1$. This shows that $P'\leqslant H\leqslant Z(G)$. So G is a GA group and $P_1'\times P_2'\times\cdots\times P_r'\leqslant H$.

Proposition 2. All subgroups and all quotient groups of a GA group are also GA groups.

Proof Suppose that G is a GA group and $H\leqslant G$. If S is a Sylow p-subgroup of H, then there exists a Sylow p Subgroup P of G such that $S\leqslant P$. Hence $S'\leqslant P'\leqslant Z(G)$. Thus, $S'\leqslant Z(H)$. So H is a GA group. Suppose that $N\trianglelefteq G$. Then every Sylow p subgroup of G/N is of form PN/N, where P is some Sylow p subgroup of G. Since $[xN, yN]=[x,y]N\in Z(G)N/N\leqslant Z(G/N)$ for x, y in P, $(PN/N)'\leqslant Z(G/N)$. Therefore, G/N is a GA group.

Proposition 3. Let G be a nilpotent group. Then G is a GA group if and only if the nilpotent class of G is at most 2, i.e., $c(G)\leqslant 2$.

Proof. This follows immediately from $G'=P_1'\times P_2'\times\cdots\times P_r'$ and the nilpotency of G.

Corollary 4. The system normalizer of G a GA group is a nilpotent group whose nilpotent class is at most 2.

We know that if G is an A-group, then $Z_\infty(G)=Z_1(G)$, where $Z_\infty(G)$ is the hypercentre of G (See [2]). In general, we have only the following.

Proposition 5. Let G be a GA group, then the upper central series of G terminates in $Z_1(G)$ or $Z_2(G)$, that is, $Z_\infty(G)=Z_2(G)$.

Proof. By the definition of GA group, $P_1'\times P_2'\times\cdots\times P_r'\leqslant Z(G)$. Then $G/Z(G)$ is an A group by proposition 1. Thus, $Z_\infty(G)/Z_1(G)=Z_\infty(G/Z_1(G))=Z_1(G/Z_1(G))=Z_2(G)/Z_1(G)$, so $Z_\infty(G)=Z_2(G)$.

Let $G=\gamma_0(G)\geqslant\gamma_1(G)\geqslant\gamma_2(G)\cdots$ be the lower central series of G, where $\gamma_i(G)=[\gamma_{i-1}(G), G]$ $(i=1,2,\cdots)$ and $\gamma_\infty(G)$ is the nilpotent residual of G. It is easy to check that $\gamma_i(G/K)=\gamma_i(G)K/K$ $(i=0,1,\cdots)$ by induction, and further $\gamma_\infty(G/K)=\gamma_\infty(G)K/K$. We know that $G'=\gamma_\infty(G)$, i.e., $\gamma_\infty(G)=\gamma_1(G)$ for A group G (See [2]). Generally, we know only the following.

Proposition 6. Let G be a GA group. Then $G'=\gamma_\infty(G)D$, where $D=P_1'\times P_2'\times\cdots\times P_r'$.

Proof By proposition 1, G/D is an A group. Then, since $D\leqslant G$ and $G'/D=(G/D)'=\gamma_\infty(G/D)=\gamma_\infty(G)D/D$, $G'=\gamma_\infty(G)D$.

Proposition 7 Let G be a GA group. Then the lower central series of G terminates in $\gamma_1(G)$ or $\gamma_2(G)$, that is, $\gamma_\infty(G)=\gamma_2(G)$.

Proof Since $G/\gamma_\infty(G)$ is nilpotent, by proposition 2 and 3, the nilpotent class $c(G/\gamma_\infty(G))\leqslant 2$. Thus $\gamma_2(G)/\gamma_\infty(G)=\gamma_2(G/\gamma_\infty(G))=1$, so $\gamma_\infty(G)=\gamma_2(G)$.

We shall study the most important properties of GA groups.

Theorem 8. Let G be a GA group and P a Sylow p subgroup of G, Then

$P\cap G'\cap Z(G)=P'$.

Proof Since G is a GA-group, $P'\leqslant Z(G)$, so that $P'\leqslant P\cap G'\cap Z(G)$. And by transfer theory, $P\cap G'\cap Z(G)<P'$ (For example, see [4]). So $P\cap G'\cap Z(G)=P'$.

Obviously, when G is an A-group, theorem 8 implies theorem A. It is easy to show that $G'\cap Z(G)=1$ is not genrally true for a GA-group G. However, we know that $\gamma_\infty(G)=G'$ for an A-group G and $\gamma_\infty(G)<G'$ for an arbitrary group G. These facts make the conjecture whetenr $\gamma_\infty(G)\cap Z(G)=1$ for a GA-group? In fact, the following so-called "special critical group" shows that the conjecture is not true. But we can prove that $\gamma_\infty(G)\cap Z(G)=1$ for a special-critical-group-free GA-group G, that is, the following theorem 10.

Definition 3. A group G is said to be special critical if it satisfies the following conditions:

(a) G has a normal extraspecial p-subgroup P for some prime p (that is, $Z(P)=P'=\Phi(P)$ and $|Z(P)|=p$);

(b) There exists a cyclic group Q of prime order q such that $G=PG$, where the prime $q\neq p$;

(c) $[P',Q]=1$, and when Q acts on P by conjugate, the induced action of Q on P/P' is fixed-point-free, and every Q-invariant proper subgroup in P is abelian. (For the extraspecial p-group and critical group, see, for example, Ⅲ, 13.7 and Ⅸ, 2.1 in [4]).

Remark. The special critical groups are separated into tow classes. When the action of Q on P/P' is irreducible, G is said to be a Ⅰ-group. Obviously, for Ⅰ-group G, the Q-invariant proper subgroups in P are only P' and 1. When the action of Q on P/P' is reducible, G is said to be Ⅱ-group. For a Ⅱ-group, by Maschke's theorem, $P/P'=H_1/P'\times\cdots\times H_s/P'$, where H_i/P' $(i=1,2,\cdots,s)$ are all irreducible Q-invariant proper subgroup of P/P' and $s>2$. We assert that $s=2$. Otherwise, whenever $i\neq j$, H_iH_j must be a Q-invariant proper subgroup of P, hence by (c), H_iH_j is abelian. However $P=H_1H_2\cdots H_s$ and is not abelian, a contradiction.

Proposition 9. Let G be a special critical group. Then (1) G is a GA-group; (2) $Z(G)=Z(P)=P'=\Phi(P)$; (3) $[P,Q]=P$ and $\gamma_\infty(G)=G'=P$, hence $\gamma_\infty(G)\cap Z(G)=Z(G)\neq 1$. (4) For erery proper section L/K of G, $\gamma_\infty(L/K)\cap Z(L/K)=1$; (5) When $p>2$, exp $P=p$; when $p=2$, exp$P=4$.

Proof. (1) Obviously.

(2) Write $Q=\langle b\rangle$, then every element of G is of form xb^i, where $x\in P$. Suppose that $xb^i\in Z(G)$. Then $b^{-1}(xb^i)b=xb^i$, hence $b^{-1}xb=x$. It follows that $x\in P'$ from the action of Q on P/P' being fixed-point-free and $Q=\langle b\rangle$. Take $y\in P$ and $y\notin P'$. Since $y(xb^i)=(xb^i)y$, by $x\in P'$ as proved above and $P'=Z(P)$, it

follows that $b^{-i}yb^{i}=y$. Again by fixed-point-free action and $y\notin P'$, it follows that $b^{i}=1$. These arguments show that $Z(G)\leqslant P'=Z(P)$. Finally, by (a) and (c), we have also $Z(P)\leqslant Z(G)$. Therefore $Z(G)=Z(P)=P'=\Phi(P)$.

(3) Consider the action of Q on P. By (c), $P'\leqslant C_p(Q)$. Conversely, if $x\in C_p(Q)$, then $b^{-1}xb=x$. However, by the action of Q on P/P' being fixed-point-free and $Q=\langle b\rangle$, it follows that $x\in P'$. Thus, $C_p(Q)=P'$. So by the theory of action of π'-groups on π-groups, it follows that $P=C_p(Q)[P,Q]=P'[P,Q]=[P,Q]$, since P' consists of non-generators of P. Thus, since $P\trianglelefteq G$ and Q is a cyclic group, $P=[P,Q]\leqslant[G,G]=[PQ,PQ]\leqslant P$, hence $G'=P$. Also $P=[P,Q]\leqslant[P,G]\leqslant P$, hence $[P,G]=P$. Thus $\gamma_\infty(G)=\gamma_2(G)=[G',G]=[P,G]=P$.

(4.1) The case where G is a Ⅰ-group. First, suppose that L is a proper subgroup of G. If L is a p-subgroup or a q-subgroup, then, obviously, $\gamma_\infty(L/K)=1$ for any quotient group L/K of L, hence $\gamma_\infty(L/K)\cap Z(L/K)=1$. Therefore, we need only to consider $pq\,\big|\,|L|$. Now $L=HQ$ where H is a Sylow p-subgroup of L, and Q is also Sylow q-subrroup of L. Since $P\trianglelefteq G$, $H\trianglelefteq L$. So H is a Q-invariant proper subgroup of P. By (c), H must be abelian. Therefore, L is an A-group. Thus, since quotient groups of an A-group are A-groups, by theorem A, it follows that $\gamma_\infty(L/K)\cap Z(L/K)=(L/K)'\cap Z(L/K)=1$ for any quotient group L/K of L.

Second, suppose that $L=G$ and $1\neq K\trianglelefteq G$. If $q\,\big|\,|K|$, then G/K is a p-group, hence, $\gamma_\infty(G/K)\cap Z(G/K)=1$. So we assume that $K\leqslant P$. Now, since KP'/P' is Q-invariant and the action of Q on P/P' is irreducible, $KP'/P'=P/P'$ or 1, hence $P=KP'=K$ or $K=P'$ (by $|P|=p$). If $K=P$, then $G/K=Q$, so that $\gamma_\infty(G/K)\cap Z(G/K)=1$ obviously; If $K=P'$, then $G/K=G/P'$, which is an A-group by proposition 1 (Note $Q'=1$), hence also $\gamma_\infty(G/K)\cap Z(G/K)=1$.

(4.2) The case where G is Ⅱ-group. First, let L and H be the same as (4.1). Since H is a Q-invariant proper subgroup of P, by (c), H is abelian. So $L=HQ$ is an A-group. Then, we immediately obtain the requared result for any quotient group of L.

Scond, suppose that $L=G$ and $1\neq K\trianglelefteq G$. By a similar to (4.1), we can assume that $K\leqslant P$, further $P'\neq K\neq P$. Obviously $P'<KP'<P$ and $KP'\trianglelefteq G$, hence, KP'/P' is a Q-invariant proper subgroup of P/P'. By Maschke's theorem, $P/P'=KP'/P'\times M/P'$, where both KP'/P' and M/P' are Q-invariant proper subgroup of P/P'. By (c), K and M must be abelian. Then, since $P=KP'M=KM$ and P is not abelian, $[K,M]\neq 1$. Thus, using $[K,M]\leqslant P'$ and $|P'|=p$, we have that $[K,M]=P'$. By $K\trianglelefteq G$, $[K,M]\leqslant K$, hence $P'\leqslant K$. Therfore, since G/P' is an A-group and $G/K=G/P'/K/P'$, G/K is also anA-group. So $\gamma_\infty(G/K)\cap Z(G/K)=1$.

(5) If $p\neq 2$, then, by $\exp P'=p$, P is a p-abelian p-group. Since $\Phi(P)=$

$Z(P)=P'=Z(G)$, it follows that if $x\in P$ and $a\in Q$, then $x^p\in Z(G)$ and hence $[x,a]^p=(x^{-1}x^a)^p=x^{-p}(x^a)^p=[x^p,a]=1$. Also, by (3) proved above, $P=[P,Q]$. Therefore, $P=\langle [x,a] \mid x\in P, a\in Q\rangle \leq \Omega_1(P)=\Lambda_1(P)\leq P$ since P is P-abelian P-group, where $\Omega_1(P)=\langle x\in P \mid x^p=1\rangle$ and $\Lambda_1(P)=\{x\in P \mid x^p=1\}$. So $P=\Lambda_1(P)$, that is, $\exp P=p$. If $p=2$. then, since P is non-abelian, $\exp P=4$.

The following is the most inportant property of GA-group, which is the main result of this paper.

Theorem 10. Let G be a GA-group and special-critical-group-free, that is, every section of G is not isomorphic to any special critical group. Then $\gamma_\infty(G)\cap Z(G)=1$.

Proof. We argue by induction on $|G|$. Let GA-group G be a counterexample to the assertion with least possible order. Through the following stepts, finally, we shall reach a contradiction, hence complete the proof.

(1) "Write $K=\gamma_\infty(G)\cap Z(G)$. Then K must be the unique minimal normal subgroup of G and of prime order p". Note that the following p, throughout the proof, refers to the prime

Obviously, $K\neq 1$ since G is a counterexample. Let $1\neq H\trianglelefteq G$. By proposition 4, G/H is also GA-group. Since G/H is also special-critical-group-free and $|G/H|<|G|$, $KH/H\leq\gamma_\infty(G)H/H\cap Z(G)H/H\leq\gamma_\infty(G/H)\cap Z(G/H)=1$. Thus, $KH=H$, hence $K\leq H$. So K is the unique normal subgroup of G. Choose arbitrarily an element a of prime order p of K. Now since $\langle a\rangle\leq K\leq Z(G)$, $\langle a\rangle\trianglelefteq G$. So $K=\langle a\rangle$ by the minimality of K.

(2) "If H is a nontrivial normal proper subgroup of G, then H must be a p-group".

Suppose that $\gamma_\infty(H)\neq 1$. Since $\gamma_\infty(H)$ char $H\trianglelefteq G$, $\gamma_\infty(H)\trianglelefteq G$- Then, by (1) proved just, $K\leq\gamma_\infty(H)$. Also, by $K\leq H$ and $K\leq Z(G)$, $K\leq Z(H)$. So $K\leq\gamma_\infty(H)\cap Z(H)$. But, since H is also special-critical-group-free and $|H|<|G|$, $\gamma_\infty(H)\cap Z(H)=1$, a contradiction. Therefore, $\gamma_\infty(H)=1$, hence H must be nilpotent. Now suppose that S is a Sylow subgroup of H. Since S char $H\trianglelefteq G$, $S\trianglelefteq G$. By (1), $K\leq S$. So H must be a p-group.

(3) "G' must be the Sylow p-subgroup of G". White $G'=P$, hence $P\trianglelefteq G$.

Since, obsiously, G is nonabelian, $G'\neq 1$. By (2) proved just, G' is a p-group. Thus, $G'\leq P$, where P is a Sylow p-subgroup of G. Obviously, G is not a p-group. Thus, if $G'\neq P$, then G/G' would have the nontrivial proper normal p-complement H/G. Hence H would be a proper normal subgroup of G and not a p-group, a contradiction to the above (2). Therefore, $G'=P$.

(4) "The p-complement Q of G must be a cyclic group of prime order q". Hence $G=PQ$ and $Q=\langle b\rangle$.

By (3), $G=G'Q$ and $G'\cap Q=1$. hence $Q\cong G/G'$ is abelian. Let H/G' be a proper subgroup of G/G'. Then H is a proper normal subgroup of G. By (2), H must be a p-group. So $H \leqslant P = G'$, hence $H/G' = 1$. This shows that G/G' (hence Q) is an abelian simple group. Therefore Q is a cyclic group of prime obder q, where $q \neq p$.

(5) "$\gamma_{\infty}(G) = P$".

Since G is a GA-group, by proposition 6 and (3) and (4), it follows that $P = G' = \gamma_{\infty}(G)P' = \gamma_{\infty}(G)$.

(6) "$Z(G) = P' = K$, in particular, $|Z(G)| = p$".

Obviously, $Z(G) \neq 1$. By (2), $Z(G)$ is a p-group, hence $Z(G) \leqslant P$. Thus, by (5), $Z(G) = Z(G) \cap P = Z(G) \cap \gamma_{\infty}(G) = K$. And by theorem 8 and (3), also $P' = P \cap G' \cap Z(G) = Z(G)$.

(7) "$Z(P)$ is a cyclic group".

Suppose the assertion to be false. $\Omega_1(Z(P)) = \langle x \in Z(P) \mid x^p = 1 \rangle$ would be an element abelian group of order p^r and $r > 1$. Since $\Omega_1(Z(P))$ char $Z(P)$ char $P \trianglelefteq G$, $\Omega_1(Z(P)) \lhd G$. By (1), $K \leqslant \Omega_1(Z(P))$. Hence $\Omega_1(Z(P)) = K \times J$. In particular $J \neq 1$ since $|K| = p$ and $r > 1$. Consider the action of Q on $\Omega_1(Z(P))$ by the conjugate. Since $K \leqslant Z(G)$, K is a Q-invariant subgroup. By Maschke's theorem, $\Omega_1(Z(P)) = K \times J_1$, where J_1 is also a Q-invariant subgroup. Hence, since $J_1 \leqslant \Omega_1(Z(P)) \leqslant Z(P)$, J_1 is invariant under the action of P by the conjugate. Then, by (1) and (2), $K \leqslant J_1$, a contradiction. Therefore $r = 1$, i.e., $Z(P)$ must be a cyclic group.

(8) "$Z(P) = P'$".

By (7), we may write $Z(P) = \langle x \rangle$, where x is of order p^n. By the property of p-groups, $n \geqslant 1$. The case: $n = 1$. Since $Z(P)$ char $P \lhd G$, $Z(P) \lhd G$. Then by (1), $K \leqslant Z(P)$. Thus $Z(P) = K$ by the orders. So $Z(P) = P'$ by (6). The case: $n > 1$. Now, $1 \neq \Omega_1(Z(P)) \lhd G$. Since (1), $K \leqslant \Omega_1(Z(P)) \leqslant Z(P)$, in particular, $K = \langle x^{p^{n-1}} \rangle$. Since $Z(P)$ char $P \lhd G$, $Z(P) \lhd G$. Thus, $b^{-1}xb = x^{\lambda}$. However, since $b^q = 1$, $\lambda^q \equiv 1 \pmod{p^n}$ by $x = b^{-q}xb^q = x^{\lambda^q}$, hence $(\lambda - 1)(\lambda^{q-1} + \cdots + \lambda + 1) \equiv 0 \pmod{p^n}$. By (6), $K = Z(G)$, hence $x^{p^{n-1}} \in Z(G)$. Thus $x^{p^{n-1}} = b^{-1}x^{p^{n-1}}b = x^{\lambda p^{n-1}}$. This shows that $p^{n-1}(\lambda - 1) \equiv 0 \pmod{p^n}$, hence $\lambda \equiv 1 \pmod{p}$. Further, it follows that $\lambda, \lambda^2, \cdots, \lambda^{q-1} \equiv 1 \pmod{p}$, so that $\lambda^{q-1} + \cdots + \lambda + 1 \equiv q \not\equiv 0 \pmod{p}$ Therefore, $\lambda \equiv 1 \pmod{p^n}$. Hence $b^{-1}xb = x$. Since $G = PQ$, $Z(P) = \langle x \rangle \leqslant Z(G)$. By (6) and the orders, we reach a contradiction. So it follows that $n > 1$ is impossible.

(9) "P is an extraspecial p-group, that is, $Z(P) = P' = \Phi(P)$ and $|Z(P)| = p$".

If x and y are in P, by (8), $x^{-1}y^{-1}xy = z \in P' = Z(P)$ and $z^p = 1$. Thus, it is easy to verify that $y^{-p}xy^p = xz^p = x$. This shows that $y^p \in Z(P)$. Hence $\mho_1(P) = \langle y^p \mid y \in P \rangle \leqslant Z(P)$. Thus $P' \leqslant \Phi(P) = P'\mho_1(P) \leqslant P'Z(P) = P'$, hence $\Phi(P) = P'$. It follows that $Z(P) = P' = \Phi(P)$ and $|Z(P)| = p$ from (6), (8).

(10) "$C_P(Q) = P'$".

Since Sylow subgroups $\bar{P} = P/P'$ and $\bar{Q} = QP'/P'$ of $\bar{G} = G/P'$ determine a Sylow basis of $\bar{G}$, $\bar{N} = N_{\bar{G}}(\bar{P}) \cap N_{\bar{G}}(\bar{Q}) = \bar{G} \cap N_{\bar{G}}(\bar{Q}) = N_{\bar{G}}(\bar{Q})$ is a system normalizer of $\bar{G}$. Then, since $\bar{G}$ is an A-group, $\bar{N}$ is a complement to $\bar{G}'$ in $\bar{G}$ (See theorem B). Clearly $\bar{N} = \bar{Q}$ by $\bar{G}' = G'/P' = P/P'$. Thus, by $\bar{Q} < \overline{N_G(Q)} < N_{\bar{G}}(\bar{Q}) = \bar{N} = \bar{Q}$, $\overline{N_G(Q)} = \bar{Q}$, that is, $N_G(Q)P' = QP'$. Also, $P' < N_G(Q)$ by (6), so that $N_G(Q) = QP'$. Thus $QP' = QZ(G) < C_G(Q) < N_G(Q) = QP'$, hence $C_G(Q) = QP'$. Now it follows that $C_P(Q) = P \cap C_G(Q) = P \cap QP' = P'$.

(11) "The action of Q on P/P' is fixed-point-free".

Suppose that $\bar{a} \in P/P'$ and $\bar{a}^b = \bar{a}$, where $Q = \langle b \rangle$ (See (4)). Then $b^{-1}ab = az$, where $z \in P' = Z(G)$. Further, $b^{-2}ab^2 = b^{-1}azb = az^2, \cdots, b^{-q}ab^q = az^q$, so that $z^q = 1$. By $(p, q) = 1$, it follows that $z = 1$, hence $b^{-1}ab = a$. Also by (10) and $Q = \langle b \rangle$, we have $a \in P'$, hence $\bar{a} = \bar{1}$.

(12) "Each of proper Q-invariant subgroups H of P is abelian".

Suppose that H is nonabelian. By $1 \neq H' < P'$ and $|P'| = p$, $H' = P'$. Consider the action of Q on H. Since $C_H(Q) = H \cap C_P(Q) = H \cap P' = H \cap H' = H'$ and $H = C_H(Q)[H, Q]$, $H = H'[H, Q] = [H, Q]$. Then, repeating the last of the argument of (3) in proposition 9, we have $\gamma_\infty(HQ) = H$. Moreover, since $Z(G) = P' = H' < H$, it follows that $Z(G) < Z(HQ)$, hence $Z(G) < \gamma_\infty(HQ) \cap Z(HQ)$. However, by the induction hypothesis and $|HQ| < |G|$, also $\gamma_\infty(HQ) \cap Z(HQ) = 1$, a contradiction.

Now, we have proved that G is a special critical group. This is contrary to the hypothesis on G. Therefore, $\gamma_\infty(G) \cap Z(G) = 1$ is true.

Let Σ be the group-theoretical property: "$\gamma_\infty(G) \cap Z(G) = 1$". Then, from proposition 9 and theorem 10, it follows that the special critical groups are exactly all of the minimal non-Σ-groups in the class of GA-groups, where so-called minimal non-Σ-group G is a group which does not have the property Σ but each of whose proper sections has the property Σ.

Finally, because an extraspecial p-group P is a central product of nonabelian subgroups of order p^3 and, conversely, a central product of nonabelian groups of order p^3 is an extraspecical p-group (See Ⅲ, 13.7 in [4]), then it is easy to construct special critical groups.

We have at once the following

Corollary 11. Let G be a GA-group. Then G is special critical-group free if and only if for every section H/K of G, $\gamma_\infty(H/K) \cap Z(H/K) = 1$.

Corollary 12. Let G be a GA-group and special-critical-group-free. Then $G' = \gamma_\infty(G) \times P_1' \times P_2' \times \cdots \times P_r'$.

The following are several applications of theorem 10. These are the generalizations of the properties A-groups (See [2], [3]) to GA-groups.

Theorem 13. Let G be a GA-group which is special-critical group-free, and let N be a system normalizer of G. Then $G=N\gamma_\infty(G)$ and $N\cap\gamma_\infty(G)=1$.

Proof. Since $N\gamma_\infty(G)/\gamma_\infty(G)$ is a system normalizer of $G/\gamma_\infty(G)$ and $G/\gamma_\infty(G)$ is nilpotent, it follows that the system normalizer of $G/\gamma_\infty(G)$ is just $G/\gamma_\infty(G)$ itself, hence $G=N\gamma_\infty(G)$. Obviously, we may assume that $\gamma_\infty(G)\neq 1$, i.e., G is not nilpotent.

Suppose that H/K is a principal factor of G and $K\lhd H\lhd\gamma_\infty(G)$. If H/K were central, i.e., $H/K\leqslant Z(G/K)$, then $H/K\leqslant\gamma_\infty(G/K)\cap Z(G/K)$. However, since G/K is also a GA-group and special-critical-group-free, $\gamma_\infty(G/K)\cap Z(G/K)=1$ by theorem 10, a contradiction. Therefore H/K must be the noncentral. Now refine the series $1<\gamma_\infty(G)<G$ into a principal series of G, i.e., $1=K_0<K_1<K_2<\cdots<K_s=\gamma_\infty(G)<\cdots<G$. Since K_i/K_{i-1} $(i=1,2,\cdots,s)$ are all noncentral, N avoids K_i/K_{i-1}, i.e., $K_i\cap N=K_{i-1}\cap N$ $(i=1,2,\cdots,s)$. Thus $\gamma_\infty(G)\cap N=K_s\cap N=\cdots=K_0\cap N=1$.

In general, we have only the following

Proporsition 14. Let G be an arbitrary GA-group and N a system normalizer of G. Then $\gamma_\infty(G)\cap N\leqslant P_1'\times P_2'\times\cdots\times P_r'$.

A special critical group G is the example such that $\gamma_\infty(G)\cap N\neq 1$.

Theorem 15. Suppose that G is a GA-group which is special-critical-group-free, and let N be a system normalizer of G and $G=L_0>L_1>\cdots>L_n=1$ the lower nilpotent series of G, where n is the nilpotent length of G. And let L be a normal subgrop of G. Then

(1) $L=(L\cap\gamma_\infty(G))(L\cap N)$ and, in particular, this is a semidirect product.

(2) If L is also nilpotent, then $L=(L\cap Z_\infty(G))\times(L\cap Z_\infty(L_1))\times\cdots\times(L\cap Z_\infty(L_n))$.

Proof (1) Let H/K be a principal factor of G. If $L\cap G'\leqslant\cdots\leqslant K<H\leqslant\cdots\leqslant L$, then $[H,G]\leqslant H\cap G'\leqslant L\cap G'\leqslant K$, so that H/K is central. If $L\cap\gamma_\infty(G)\leqslant\cdots\leqslant K<H\leqslant\cdots\leqslant L\cap G'$, then, since G is a GA-group and by proposition 7, $[H,G]\leqslant[G',G]=\gamma_2(G)=\gamma_\infty(G)$, hence $[H,G]\leqslant H\cap\gamma_\infty(G)\leqslant L\cap\gamma_\infty(G)\leqslant K$, so that H/K is also central. Now, refine the series $L\cap\gamma_\infty(G)\leqslant L\cap G'\leqslant L\leqslant G$ into a pricipal series $1\leqslant\cdots\leqslant L\cap\gamma_\infty(G)=K_0<K_1<\cdots\leqslant L\cap G'\leqslant\cdots<K_s=L\leqslant\cdots\leqslant G$. Since all pricipal factors K_i/K_{i-1} $(i=1,2,\cdots,s)$ are central, the system normalizer N covers K_i/K_{i-1}, i.e., $NK_i=NK_{i-1}$ $(i=1,2,\cdots,s)$. Thus $NL=N(L\cap\gamma_\infty(G))$. Then $L=L\cap(NL)=L\cap[N(L\cap\gamma_\infty(G))]=(L\cap\gamma_\infty(G))(L\cap N)$. Form theorem 13, it follows that the product is a semidirect product.

(2) Let $\{P_1,P_2,\cdots,P_r\}$ be a sylow basis corresponding to the system normalizer N, i.e., $N=\prod_{i=1}^{r}N_G(P_i)$, hence $N\leqslant N_G(P_i)$ $(i=1,2,\cdots,r)$. Thus $[L\cap N,P_i]$

$\leqslant P_i$. Since $L \lhd G$ and L is nilpotent, it follows that $L_{p_i'} \trianglelefteq G$ and $L_{p_i} \lhd G$, where $L_{p_i'}$ is the p_i-complement of L and L_{p_i} is the Sylow p_i-subgroup of L. So $L_{p_i} \lhd P_i$. Also, if $a \in L \cap N$ and $z \in P_i$, then $a = xy$, where $x \in L_{p_i}$ and $y \in L_{p_i'}$. Thus, $[a, z] = [xy, z] = [x, z]^y [y, z] = [x, z][y, z] \in P_i' L_{p_i'}$ by $P_i' \leqslant Z(G)$. Therefore, $[L \cap N, P_i] \leqslant P_i \cap (P_i' L_{p_i'}) = P_i' (P_i \cap L_{p_i'}) = P_i'$. Further, for $a \in L \cap N$ and $z \in G$, we have $z = z_1 z_2 \cdots z_r$, where $z_i \in P_i$ $(i = 1, 2, \cdots, r)$. By $[L \cap N, P_i] \leqslant P_i' \leqslant Z(G)$, $[a, z] = [a, z_1 z_2 \cdots z_r] = [a, z_2 \cdots z_r][a, z_1]^{z_2 \cdots z_r} = [a, z_2 \cdots z_r][a, z_1] = \cdots = [a, z_r] \cdots [a, z_2][a, z_1] \leqslant Z(G)$. Thus $[L \cap N, G] \leqslant Z(G)$. So it follows that $L \cap N \leqslant Z_2(G)$, hence $L \cap N \leqslant L \cap Z_\infty(G)$ by proposition 5. Since $Z_\infty(G) = \mathrm{Core}_G(N)$ (See Ⅵ. 11 in [4]), $L \cap N \leqslant L \cap Z_\infty(G) = L \cap \mathrm{Core}_G(N) \leqslant L \cap N$, hence $L \cap N = L \cap Z_\infty(G)$. Now $[\gamma_\infty(G), Z_\infty(G)] \leqslant \gamma_\infty(G) \cap Z_\infty(G) = \gamma_\infty(G) \cap \mathrm{Core}_G(N) \leqslant \gamma_\infty(G) \cap N = 1$ by theorem 13. So $\gamma_\infty(G)$ commutes with $Z_\infty(G)$ element-wise. Sum up the above facts, and by (1) proved just, it follows that $L = (L \cap \gamma_\infty(G)) \times (L \cap Z_\infty(G))$. Since $L \cap \gamma_\infty(G)$ is a nilpotent normal subgroup of $\gamma_\infty(G)$, $L \cap \gamma_\infty(G) = [(L \cap \gamma_\infty(G)) \cap \gamma_\infty(\gamma_\infty(G))] \times [(L \cap \gamma_\infty(G)) \cap Z_\infty(\gamma_\infty(G))] = [L \cap \gamma_\infty(\gamma_\infty(G))] \times [L \cap Z_\infty(\gamma_\infty(G))] = (L \cap L_2)(L \cap Z_\infty(L_1))$, hence $L = (L \cap Z_\infty(G)) \times (L \cap Z_\infty(L_1)) \times (L \cap L_2)$. the required result follows from repeating the process.

Corollary 16 If G is a GA-group and special-critical-group-free, then Fitting subgroup $F(G) = Z_\infty(L_0) \times Z_\infty(L_1) \cdots \times Z_\infty(L_n)$.

This theorem is not necessary true for an arbitrary GA-group, for example, the special critical groups.

Theorem 17 Let G be a GA-group which is special-critical -group-free and D a system normalizer of G. Then $N_G(D) = D \times (\gamma_\infty(G) \cap C_G(D))$.

Proof By theorem 13, $G = D\gamma_\infty(G)$ and $D \cap \gamma_\infty(G) = 1$, hence $N_G(D) = N_G(D) \cap (D\gamma_\infty(G)) = D(\gamma_\infty(G) \cap N_G(D))$. However, $[D, \gamma_\infty(G) \cap N_G(D)] \leqslant D \cap \gamma_\infty(G) = 1$, i .e., $\gamma_\infty(G) \cap N_G(D)$ commutes with D elementwise. Thus $\gamma_\infty(G) \cap N_G(D) \leqslant \gamma_\infty(G) \cap C_G(D)$. Then, $\gamma_\infty(G) \cap C_G(D) = \gamma_\infty(G) \cap N_G(D)$ by $\gamma_\infty(G) \cap C_G(D) \leqslant \gamma_\infty(G) \cap N_G(D)$. Therefore, $N_G(D) = D \times (\gamma_\infty(G) \cap C_G(D))$.

An A-group G, obviously, is a GA-group which is special-critical-group-free. And $\gamma_\infty(G) = G'$ and $Z_\infty(G) = Z(G)$. So the above theorems imply the corresponding theorems on A-groups.

References

[1] P.Hall, J.Math. 182 (1940), 206—14.

[2] D. R. Taunt, Proc. Combridge Phil . Soc. 45, 14—42 (1949).

[3] R . W. Carter, Proc. London Math. Soc. 12, 535—563(1962).

[4] B.Huppert , Endliche Gruppen, Springer, 1967.

关　于　广　A—群

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摘　　要

P.Hall定义了A-群的概念，所谓A-群是指Sylow子群都是交换群的有限可解群．D.R.Taunt，B.Huppert 和R.W.Carter 都研究过A-群的结构．本文定义了比A-群更弱的一类群，即广A-群(或称GA-群)，并将A-群的若干重要性质推广到广A-群．

定义 1　称有限可解群G是广A-群，如果G的所有Sylow子群的导群都属于G的中心。

为了描述广A-群的性质，我们还引入了

定义 2　称群G为特殊临界群，如果（a）．G有一个正规的超特殊p-子群P（即$Z(P)=P'=\Phi(P)$且$|Z(P)|=p$)；(b)．存在素数$q(\neq p)$阶子群Q，使$G=PQ$；（c)．$[P',Q]=1$，Q依共轭无不动点地作用于P/P'且P的任意Q-不变真子群都是交换群．

可证，对于特殊临界群G，G必为广A-群，且$\gamma_\infty(G)\cap Z(G)\neq 1$，但对$G$的每个真截断(section)$H/K$，有$\gamma_\infty(H/K)\cap Z(H/K)=1$，这里$\gamma_\infty(G)$表示$G$的幂零剩余．

本文的主要结果是：

定理　设G是广A-群且每个截断都不是特殊临界群(也称G与特殊临界群无关)，则$\gamma_\infty(G)\cap Z(G)=1$．

我们还给出了广A-群上述基本性质的几个应用．

设G是广A-群且与特殊临界群无关，又令N是G的一个系正规化子(system normalizer)，那么，

（1）$G'=\gamma_\infty(G)\times P_1'\times\cdots\times P_r'$，这里，$P_1,\cdots,P_r$是$G$的不同素因子对应的Sylow子群．

（2）若L为G的正规子群，则$L=(L\cap\gamma_\infty(G))(L\cap N)$且是一个半直积，特别地，$G=N\gamma_\infty(G)$且$N\cap\gamma_\infty(G)=1$．

（3）若L为G的正规的幂零子群，又设$G=L_0>L_1>\cdots>L_n=1$是G的下幂零群列，n为幂零长，则$L=(L\cap Z_\infty(G))\times(L\cap Z_\infty(L_1))\times\cdots\times(L\cap Z_\infty(L_n))$，这里，$Z_\infty(G)$表示$G$的超中心．

（4）$N_G(N)=N\times(\gamma_\infty(G)\cap C_G(N))$.

最后，当G为A-群时，上述结果即为A-群的已知结论．