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Characterizations of Inverse p.n.p. Matrices*

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Abstract

The author studies the characterizations of inverse p.n.p. matrices, i.e., theirs inverse matrices are p.n.p. matrices.

Recall that an n-by-n real matrix A is called a partially nonpositive (p.n.p.) matrix if each of its principal minors is nonpositive and denote $A\in$ PNP; if each of its principal minors is negative then A is called a partially negative (p.n.) matrix and denote $A\in$PN. In[1-4] disscused some properties and characterizations of p.n.p. matrices. In this paper we introduce the concept of inverse p.n.p. matrices, disscuss its equivalent characterizations and give the structure of the intersection set of p.n.p. matrices and inverse p.n.p. matrices.

Definition Let $A=(a_{ij})\in R^{n\times n}$ be nonsingular. If $A^{-1}\in$PNP, then A is called an inverse p.n.p. matrix and denote $A\in$IPNP; if $A^{-1}\in$PN, then A is called an inverse p.n. matrix and denote $A\in$IPN.

Lemma 1 [4] Let $A=(a_{ij})\in R^{n\times n}$ be nonsingular. Then $A\in$IPNP (IPN) if and only if $\det A<0$, $\det A_k\geqslant(>)0$, where $\det A_k$ denote the principal minor k, $1\leqslant k\leqslant n-1$.

Lemma 2 Let $A=(a_{ij})_{n\times n}\in$IPNP, then there exists a positive number ε_0 such that $A+\varepsilon I\in$IPN when $0<\varepsilon<\varepsilon_0$.

Proof. Denote the sum of principal minors of orders i for matrix $B=A^{-1}$ by $-\tilde{s}_i\,(i=1,\cdots,n)$, and the sum of principal minors j for matrix A by s_j $(j=1,\cdots,n-1)$. By [3] we have that $-\rho(B)$ is a simple eigenvalue of B and satisfies

$$\rho(B)\leqslant\min\{\max(1,\sum_{i=1}^{n}\tilde{s}_i),\ \max(\tilde{s}_n,\tilde{s}_{n-1}+1,\cdots,\tilde{s}_1+1). \qquad (1)$$

Thus $\sigma=\frac{-1}{\rho(B)}$ is a simple eigenvalue of A. Since $-\tilde{s}_i=\frac{s_{n-i}}{\det A}$ $(i=1,\cdots,n-1)$ ([4]), if denote $s_n=1$, by (1) we have

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$$-\sigma=|\sigma|\geqslant[\min\{\max(1,-\sum_{i=1}^{n}s_i/\det A),\max(\frac{-1}{\det A},1-\frac{s_{n-1}}{\det A},\cdots,$$

$$1-\frac{s}{\det A})\}]^{-1}=\frac{-1}{\det A}[\min\{\max(-\det A,\sum_{i=1}^{n}s_i),\max(1,s_{n-1}-\det A,\cdots,$$

$$s_1-\det A)\}]^{-1}>0. \qquad (2)$$

Take $\varepsilon_0'=\frac{-1}{\det A}[\min\{\max(-\det A,\sum_{i=1}^{n}s_i),\max(1,s_{n-1}-\det A,\cdots,s_1-\det A)\}]^{-1}$, and denote the spectrum of A by $\sigma(A)=\{\lambda_i\,|i=1,\cdots,n\}$ and $\lambda_1=\sigma$, then $|\lambda_i|>|\lambda_1|$, $i=2,\cdots,n$. Moreover let $\lambda_1,\cdots,\lambda_r$ and $\lambda_{r+1},\cdots,\lambda_n$ be real and complex eigenvalues of A, respectively. Thus $\lambda_i+\varepsilon$ and λ_i have same signs $(i=1,\cdots,r)$ when $0<\varepsilon<\varepsilon_0'$, so $\det(A+\varepsilon I)=\prod_{i=1}^{n}(\lambda_i+\varepsilon)$ and $\det\ A=\prod_{i=1}^{n}\lambda_i$ have same sign, i.e., $\det(A+\varepsilon I)<0$.

Let A_{kj} be any principal submatrix of A of order k $(k=1,\cdots,n-1;j=1,\cdots,c_n^k)$, and denote greatest negative eigenvalue of A_{kj} by $-\varepsilon_{kj}$(if greatest negative eigenvalue of A_{kj} vanish let ε_{kj} be any positive number). Then $A_{kj}\epsilon P_0$ by Lemma 1, where P_0 denotes the set of matrices with nonnegative principal minors ([5]). Therefore det $(A_{kj}+\varepsilon I_k)=\prod_{i=1}^{k}(\lambda_i^{(kj)}+\varepsilon)>0$ for any $0<\varepsilon<\varepsilon_{kj}$, where $\{\lambda_i^{(kj)}\,|i=1,\cdots,k\}$ denotes the spectrum of A_{kj}, so $A_{kj}+\varepsilon I_k\epsilon$P ($P$ denotes the set of matrices with positive principal minors). Take $\varepsilon_0=\min\{\varepsilon_0',\varepsilon_{11},\cdots,\varepsilon_{1c_n^1},\cdots,\varepsilon_{n-11},\cdots,\varepsilon_{n-1c_n^{n-1}}\}$, then $A+\varepsilon I\epsilon$IPN for any $0<\varepsilon<\varepsilon_0$ by Lemma 1.

Theorem 1 Let $A=(a_{ij})\epsilon R^{n\times n}$ be nonsingular. Then $A\epsilon$IPNP if and only if there exists *a* positive ε_0 such that $A+\varepsilon I\epsilon$IPN when $0<\varepsilon<\varepsilon_0$.

Proof Only prove the sufficiency by Lemma 2. Using marks of Lemma 2 for sufficient small $\varepsilon>0$ we have det $(A+\varepsilon I)=\prod_{i=1}^{n}(\lambda_i+\varepsilon)<0$. Since A is nonsingular hence det $A<0$ when $\varepsilon\to0$. Similarly for any principal submatrix A_k of A $(k=1,\cdots,n-1)$ we have det $A_k\geqslant0$. If not, there exists $1\leqslant k_0\leqslant n-1$ such that det $A_{k_0}<0$, since $A_{k_0}\epsilon R^{k_0\times k_0}$ so A_{k_0} has at least one negative eigenvalue, denote the greatest negative eigenvalue of A_{k_0} by $\lambda_1^{(k_0)}$, denote its spectrum by $\sigma(A_{k_0})=\{\lambda_i^{(k_0)}\,|i=1,\cdots,k_0\}$. Take $0<\varepsilon\leqslant|\lambda_1^{(k_0)}|$, then det $(A_{k_0}+\varepsilon I_{k_0})=\prod_{i=1}^{k_0}(\lambda_i^{(k_0)}+\varepsilon)<0$ for any $0<\varepsilon<\varepsilon_0$, which is a contradiction. Thus $A\epsilon$IPNP by Lemma 1.

Let $A=(a_{ij})\epsilon R^{n\times n}$ be partitioned as

$$A=\begin{pmatrix}A_k & A_1\\ A_2 & A_{n-k}\end{pmatrix},\quad 1\leqslant k\leqslant n-1, \qquad (3)$$

where A_k is the leading principal subratrix of A of order k. If A_k is nonsingular then $A_{n-k}-A_2A_k^{-1}A_1$ is called the Schur complement of A_k in A, we denote this $(A/A_k)=A_{n-k}-A_2A_k^{-1}A_1$.

Theorem 2 Let $A=(a_{ij})\in R^{n\times n}$ be partitioned as in (3). Then $A\in$IPN if and only if det $A<0$, $A_k\in$P and $(A/A_k)\in$IPN.

Proof (⇒) Only prove $(A/A_k)\in$IPN by Lemma 1. Let $L=\begin{pmatrix} I_k & -A_k^{-1}A_1 \\ 0 & I_{n-k}\end{pmatrix}$ then

$$\begin{pmatrix} A_k & A_1 \\ A_2 & A_{n-k}\end{pmatrix}\begin{pmatrix} I_k & -A_k^{-1}A_1 \\ 0 & I_{n-k}\end{pmatrix}=\begin{pmatrix} A_k & 0 \\ A_2 & (A/A_k)\end{pmatrix}=\tilde{A} \qquad (4)$$

Thus

$$\det\tilde{A}=\det A\cdot\det L=\det A=\det A_k\cdot\det(A/A_k).$$

Since det $A<0$, det $A_k>0$, hence det$(A/A_k)<0$. Let $(A/A_k)_s$ be the leading principal submatrix of (A/A_k) of order s $(s=1,\cdots,n-k-1)$ and L be partitioned as $L=\begin{pmatrix} L_{k+s} & L_1 \\ 0 & L_{n-k-s}\end{pmatrix}$ where L_{k+s} is the leading principal submatrix of L of order $k+s$. Moreover Let A and $\tilde{A}$ be partitioned as

$$A=\begin{pmatrix} A_{k+s} & A_1' \\ A_2' & A_{n-k-s}\end{pmatrix}, \qquad \tilde{A}=\begin{pmatrix} \tilde{A}_{k+s} & 0 \\ \tilde{A}_2 & \tilde{A}_{n-k-s}\end{pmatrix},$$

by (4) we have

$$\begin{pmatrix} A_{k+s} & A_1' \\ A_2' & A_{n-k-s}\end{pmatrix}\begin{pmatrix} L_{k+s} & L_1 \\ 0 & L_{n-k-s}\end{pmatrix}=\begin{pmatrix} \tilde{A}_{k+s} & 0 \\ \tilde{A}_2 & \tilde{A}_{n-k-s}\end{pmatrix}. \qquad (5)$$

So $\tilde{A}_{k+s}=A_{k+s}L_{k+s}$ and det $\tilde{A}_{k+s}=\det A_{k+s}>0$. Since det $\tilde{A}_{k+s}=\det A_k\det(A/A_k)_s$ and det $A_k>0$ hence det $(A/A_k)_s>0$. In general if $(A/A_k)_s$ denotes principal submatrix of (A/A_k) of order s $(s=1,\cdots,n-k-1)$, then there exists a permutation matrix Q_{n-k} of order $n-k$ such that $(A/A_k)_s$ is the leading principal submatrix of $Q_{n-k}(A/A_k)Q_{n-k}$ of order s. Let $Q=\begin{vmatrix} I_k & 0 \\ 0 & Q_{n-k}\end{vmatrix}$ then Q is a permutation matrix of order n. By (4) and $QAQ\cdot QLQ=Q\tilde{A}Q$ we have

$$\begin{pmatrix} I_k & 0 \\ 0 & Q_{n-k}\end{pmatrix}\begin{pmatrix} A_k & A_1 \\ A_2 & A_{n-k}\end{pmatrix}\begin{pmatrix} I_k & 0 \\ 0 & Q_{n-k}\end{pmatrix}\begin{pmatrix} I_k & 0 \\ 0 & Q_{n-k}\end{pmatrix}\begin{pmatrix} I_k & -A_k^{-1}A_1 \\ 0 & I_{n-k}\end{pmatrix}\begin{pmatrix} I_k & 0 \\ 0 & Q_{n-k}\end{pmatrix}$$
$$=\begin{pmatrix} A_k & 0 \\ Q_{n-k}A_2 & Q_{n-k}(A/A_k)Q_{n-k}\end{pmatrix}. \qquad (6)$$

Clearly $QAQ^{\mathrm{T}}=QAQ\in$IPN and $QLQ^{\mathrm{T}}=QLQ$ is also elementary matrix, hence det $(Q_{n-k}(A/A_k)Q_{n-k})_s=\det(A/A_k)_s>0$ where $(Q_{n-k}(A/A_k)Q_{n-k})_s$ denotes the leading principal submatrix of $Q_{n-k}(A/A_k)Q_{n-k}$ of order s $(s=1,\cdots,n-k-1)$. Therefore we deduce $(A/A_k)\in$IPN by Lemma 1.

(⇐). Since $A_k\in$P hence det $A_t>0$ where A_t denotes the principal submatrix of A of order t $(t=1,\cdots,k)$. For any principal submatrix A_{k+s} of order

$k+s$ $(s=1,\cdots,n-k-1)$ since det $(A/A_k)_s>0$ and det $A_k>0$ we have det $A_{k+s}>0$ by (4) and (5). Thus $A\in$IPN by Lemma 1 and det $A<0$.

Similarly may prove

Corollary 1 Let $A=(a_{ij})\in R^{n\times n}$ be nonsingular. If A is partitioned as (3) and A_k is also nonsingular $(1\leqslant k\leqslant n-1)$, then $A\in$IPNP if and only if det $A<0$, $A_k\in P_0$, $(A/A_k)\in$IPNP.

Corollary 2 Let $A=(a_{ij})\in R^{n\times n}$. Then $A\in$PN if and only if $a_{ii}<0$, $(A/a_{ii})\in P$ and det $A<0$, $i=1,\cdots,n$.

Combined Theorem 1 in [4] we can deduce sufficient and necessary condition for $A\in$PNP which is omited.

Corollary 3 Let $A=(a_{ij})\in R^{n\times n}$ be nonsingular. Then $A\in$IPNP if and only if there exists a positive number ε_0 such that det $(A+\varepsilon I)<0$, $A_k+\varepsilon I_k\in P$ and $(A+\varepsilon I/A_k+\varepsilon I_k)\in$IPN when $0<\varepsilon<\varepsilon_0$, where A is partitioned as in form (3), $1\leqslant k\leqslant n-1$.

Finally, we give the structure of the intersection set of p.n.p. matrices and inverse p.n.p. matrices. Clearly we have following Lemma.

Lemma 3 Let $A=(a_{ij})\in R^{n\times n}$ be nonsingular. Then $A\in$PNP$\cap$IPNP if and only if det $A<0$, det $A_k=0$, where A_k is any principal submatrix of order k of A, $k=1,\cdots,n-1$.

Theorem 3 Let $A=(a_{ij})\in R^{n\times n}$ be nonsingular. Then $A\in$PNP$\cap$IPNP if and only if there exists a permutation matrix Q of order n such that

$$QAQ^{\mathrm{T}}=B=(b_{ij})_{n\times n}=\begin{pmatrix} 0 & b_{12} & 0 & \cdots & 0 \\ 0 & 0 & b_{23} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & & \ddots & b_{n-1\,n} \\ b_{n1} & 0 & \cdots & \cdots & 0 \end{pmatrix}, \tag{7}$$

where $(-1)^n b_{12}b_{23}\cdots b_{n-1\,n}b_{n1}<0$.

Proof We need only prove necessity. Since A is nonsingular and irreducible ([4]), by Lemma 3 we have that there exists a shortest circuit with length n in the directed graph $G(A)$ of A (or else there exists a circuit γ with leng k in the directed graph $G(A_k)$ of some principal submatrix A_k and there exist no other circuits in $G(A_k)$, $2\leqslant k\leqslant n-1$; thus det $A_k\neq 0$, which is a contradiction). Therefore without loss of generality we assume that A has the following form:

$$A=\begin{pmatrix} 0 & a_{12} & * & \cdots & * \\ * & 0 & a_{23} & & \vdots \\ \vdots & & & & * \\ * & & & & a_{n-1\,n} \\ a_{n1} & * & & * & 0 \end{pmatrix} \tag{8}$$

where $a_{12}a_{23}\cdots a_{n-1\,n}a_{n1}\neq 0$. Since all principal minors of order 2 of A ore zero hence $a_{21}=a_{32}=\cdots=a_{n\,n-1}=a_{1n}=0$. Denote the principal submatrix of A with $i_1,\cdots,i_k$ rows and $i_1,\cdots,i_k$ columns by $A[i_1,\cdots,i_k]$ where $1\leq i_1<i_2<\cdots<i_k\leq n$. Therefore from $\det A[1,2,3]=\det[2,3,4]=\cdots=\det[n-2,n-1,n]=0$ deduce $a_{31}=a_{42}=\cdots=a_{n\,n-2}=0$; from $\det A[1,2,3,4]=\det A[2,3,4,5]=\cdots=\det A[n-3,n-2,n-1,n]=0$ deduce $a_{41}=a_{52}=\cdots=a_{n\,n-3}=0$;$\cdots$; from $\det A[1,\cdots,n-1]=\det A[2,\cdots\cdots,n=0$ deduce $a_{n-1\,1}=a_{n2}=0$. That is $a_{ij}=0$ when $j<i$ except a_{n1}.

Moreover from $a_{1n}=0$ and $\det A[1,2,n]=\det A[1,2,3,n]=\cdots=\det A[1,2,\cdots,n-2,n]=0$ deduce $a_{2n}=a_{3n}=\cdots=a_{n-2\,n}=0$; from $\det A[1,n-1,n]=\det A[1,2,n-1,n]=\cdots=\det A[1,\cdots,n-3,n-1,n]=0$ deduce $a_{1\,n-1}=a_{2\,n-1}=\cdots=a_{n-3\,n-1}=0$; from $\det A[1,n-2.n-1,n]=\det A[1,2,n-2,n-1,n]=\cdots=\det A[1,\cdots,n-4,n-2,n-1,n]=0$ deduce $a_{1\,n-2}=a_{2\,n-2}=\cdots=a_{n-4\,n-2}=0$;$\cdots$; from $\det A[1,3,\cdots,n]=0$ deduce $a_{13}=0$.

In a word we have that A has form (7), and since $\det A<0$ hence $(-1)^{n-1}a_{12}a_{23}\cdots a_{n-1\,n}a_{n1}<0$.

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逆 p.n.p. 矩阵的表征

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摘 要

一个 n 阶实方阵 A，若其各阶主子式皆非正，则称 A 为 p.n.p. 矩阵，记作 $A\in$PNP；特别地，若 $A\in$PNP 且各阶主子式皆负，则称 A 为 p.n. 矩阵，记作 $A\in$PN 进一步，若 n 阶实方阵 A 非奇异，且 $A^{-1}\in$PNP，则称 A 为逆 p.n.p. 矩阵，记作 $A\in$IPNP；特别地，若 $A^{-1}\in$PN，则称 A 为逆 p.n. 矩阵，记作 $A\in$IPN．本文的主要结果如下：

定理 1 设 $A\in R^{n\times n}$ 为非奇异．则 $A\in$IPNP 的充要条件是存在一个正数 ε_0，使得当 $0<\varepsilon<\varepsilon_0$ 时，总有 $A+\varepsilon I\in$IPN．

定理 2 设 $A\in R^{n\times n}$ 分块为

$$A=\begin{vmatrix} A_k & A_1 \\ A_2 & A_{n-k} \end{vmatrix}, \qquad (*)$$

其中 A_k 为 A 之 k 阶主要主子阵（$1\leqslant k\leqslant n-1$）．则 $A\in$IPN 的充要条件是 $\det A<0$，$A_k\in$P（P 表所有主子式皆正的矩阵类）且 $(A/A_k)\in$IPN，这里 (A/A_k) 表 A_k 于 A 中的 Schur 补．

推论 1 设 $A\in R^{n\times n}$ 分块形如（$*$），且 A 与 A_k 皆非奇异．则 $A\in$IPNP 的充要条件是 $\det A<0$，$A_k\in P_0$（P_0 表所有主子式皆非负的矩阵类）且 $(A/A_k)\in$IPNP．

推论 2 设 $A=(a_{ij})\in R^{n\times n}$．则 $A\in$PN 的充要条件是 $a_{ii}<0$ 且 $(A/a_{ii})\in$P，$\det A<0$ $(1\leqslant i\leqslant n)$．

推论 3 设 $A\in R^{n\times n}$ 为非奇异．则 $A\in$IPNP 的充要条件是存在一个正数 ε_0，使得当 $0<\varepsilon<\varepsilon_0$ 时，总有 $\det(A+\varepsilon I)<0$，$A_k+\varepsilon I_k\in$P 且 $(A+\varepsilon I/A_k+\varepsilon I_k)\in$IPN，其中 A 分块如（$*$），$1\leqslant k\leqslant n-1$，

定理 3 设 $A\in R^{n\times n}$ 为非奇异，则 $A\in$PNP$\cap$IPNP 的充要条件是存在一个 n 阶置换阵 Q 使得

$$QAQ^{\mathrm{T}}=B=(b_{ij})_{n\times n}=\begin{vmatrix} 0 & b_{12} & 0 & \cdots & 0 \\ \vdots & 0 & b_{23} & & 0 \\ 0 & \vdots & & & b_{n-1\,n} \\ b_{n1} & 0 & \cdots\cdots & & 0 \end{vmatrix},$$

其中 $(-1)^{n-1}b_{12}b_{23}\cdots b_{n-1\,n}b_{n1}<0$。