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Remarks on the Asymptotic Normality of Least Squares Estimator for System Identification*

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Abstract

In Yuan's paper [1], we have proved the asymptotic normality of least square estimator in system identification using the central limit theorem for martingales. However, the conditions of [1] are rather harsh. In this artical, we use Mcleish's dependent central limit theorem to improve the above result.

Rest Restatement of the Problem

Consider the following system in regressive form

$$y(t)=\phi^T(t)\theta+(t) \tag{1.1}$$

where

$$\theta^T=(a_1 a_2\cdots a_{n_a}\ b_1\cdots b_{n_b})\quad \text{to be estimated}$$
$$\phi^T(t)=[-y(t-1)\cdots-y(t-n_a),\ u(t-1),\cdots,\ u(t-n_b)],$$

$u(t)$ and $y(t)$ are the scalar input and output at time t.

Asumptions on the system and noise:

C1: The system (1.1) is strictly causal, i.e., $n_a>n_b$. Morever, $y(t)=u(t)=0$, when $t<0$.

C2: The all zeros of $A(q^{-1})$ are strictly inside the unit circle.

C3: $\{\varepsilon(t)\}_{t=0}^{\infty}$ is a martingale difference sequence. If $\mathcal{F}_{n,t}$ denotes the smallest σ-algebra generated by $\varepsilon(0)$, $\varepsilon(1),\cdots\varepsilon(t)$, then $E[\varepsilon(t)|\mathcal{F}_{n,t-1}]=0$ and $E[\varepsilon^2(t)|\mathcal{F}_{n,t-1}]=\Lambda_t^2$ for any $t=1,2,\cdots,n$.

C4: The 4th moments of $\varepsilon(t)$ is finite, i.e.,

$$E[\varepsilon^4(t)|\mathcal{F}_{n,t-1}]=\beta_t^4<\infty \text{ for any } t=0,1,2,\cdots \tag{1.2}$$

As usual, we assume that $u(t)$ and $y(t)$ are $\mathcal{F}_{n,t}$ measurable, then the following proof will include the closed-loop case.

The criterion function is

$$J_n=\frac{1}{n}\sum_{t}^{n}{}_{1} M_t(Y(t)-\varphi^T(t)\theta)^2 \tag{1.3}$$

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where M_t is a weighting function,

The LS estimator of θ is

$$\hat{\theta}=P_n[\frac{1}{n}\sum_{t=1}^{n} M_t\phi(t)Y(t)] \tag{1.4a}$$

$$P_n^{-1}=\frac{1}{n}\sum_{t=1}^{n} M_t\phi(t)\phi^T(t) \tag{1.4b}$$

Introduce some notations,

$$S_{n,k}=\frac{1}{\sqrt{n}}\sum_{t=1}^{n} M_t\phi(t)\varepsilon(t) \tag{1.5}$$

With some calculus, we obtain that

$$S_{n,k}=\sqrt{n}\,P_n^{-1}(\hat{\theta}_n-\theta) \tag{1.6}$$

The problem is to discuss the asymptotic distributton of $S_{n,k}$.

Main Results

In order to obtain the asymptotic distribution of $S_{n,k}$, we need the following lemmas.

Lemma 1. If C1 and C2 are held

$$|Eu(t_1)u(t_2)u(t_3)u(t_4)|<r<\infty,\quad \forall t_1,t_2,t_3,t_4\in \boldsymbol{N} \tag{2.1}$$

where r is a constant, then

$$E\bar{y}^4(t)<M_1<\infty \tag{2.2}$$

here $A(q^{-1})\bar{y}(t)=B(q^{-1})u(t)$ and M_1 is a constant .

Proof. From condition C1, we have

$$y(t)=\sum_{t=1}^{t-1} g_i u(t-i)$$

Hence $$E\bar{y}^4(t)=|E\sum_{i_1,i_2,i_3,i_4=1}^{t-1} g_{i_1},g_{i_2},g_{i_3},g_{i_4}u(t-i_1)u(t-i_2)u(t-i_3)u(t-i_4)|$$

$$<r(\sum_{i=1}^{t-1} g_i)^4<r(\sum_{i=1}^{\infty} g_i)^4<\infty$$

Lemma 2. If C1, C2, C3, and C4 are held and

$$E\bar{\varepsilon}^4(t)=\beta_t^4<W<\infty \tag{2.3}$$

where W is a constant, then

$$E\bar{\varepsilon}^4(t)<M_2 \tag{2.4}$$

where $A(q^{-1})\bar{\varepsilon}(t)=\varepsilon(t)$ and M_2 is a constant .

The proof is similar to that of lemma 1.

Lemma 3. If C1, C2, C3, C4, (2.1) and (2.3) are satisfied, then $Ey^4(t)<M$, $\forall t>0$

where M is a constant .

Proof. From (1.1), we have

$$y^4(t)=(\frac{B(q^{-1})}{A(q^{-1})}u(t))^4+4(\frac{B(q^{-1})}{A(q^{-1})}u(t))^3(\frac{1}{A(q^{-1})}\varepsilon(t))+6(\frac{B(q^{-1})}{A(q^{-1})}u(t))^2\cdot$$

$$\cdot(\frac{1}{A(q^{-1})}\varepsilon(t))^2+4(\frac{B(q^{-1})}{A(q^{-1})}u(t))(\frac{1}{A(q^{-1})}\varepsilon(t))^3+(\frac{1}{A(q^{-1})}\varepsilon(t))^4$$

According to Lemma 1, Lemma 2 and Hölder inequalily, we come to the conclusion:

Henceforth if x is a vector (x_i is the ith element of x), x denotes the vector in which the ith element is x_i.

Lemma 4. If C1, C2, C3, C4, (2.1) and (2.3) are satisfied and $\sup M_t<\delta<\infty$ then,

$$\max_{t\le n}|\frac{1}{\sqrt{n}}\phi(t)M_t\varepsilon(t)|\xrightarrow{P}0,\ \text{as},\ n\to\infty \tag{2.5}$$

here P denotes convergence in probability.

Proof. Let $\omega_n=\max_{t\le n}|\frac{1}{\sqrt{n}}u(t-l)M_t\varepsilon(t)|$.

then for $\lambda>0$

$$P(\omega'_n>\lambda)\le\sum_{t=1}^{n}P(|\frac{1}{\sqrt{n}}u(t-l)M_t\varepsilon(t)|>\lambda)$$

$$<\frac{\delta^4}{\lambda^4n^2}\sum_{t=1}^{n}Eu^4(t-l)\beta_t^4<\frac{\delta^4}{\lambda^4n^2}\sum_{t=1}^{n}\beta_t^4 \longrightarrow 0,\ \text{as}\ n\to\infty$$

The second inequality helds because of C3, C4 and Markov's inequality. Hence,

$$\omega'_n\xrightarrow{P}0,\ \text{as}\ n\to\infty,\ \ l=1,2,\cdots,n_b \tag{2.6}$$

Let $$r_n^j=\max_{t\le n}|\frac{1}{\sqrt{n}}y(t-j)M_t\varepsilon(t)|$$

$$P(r_n^j>\lambda)<\sum_{t=1}^{n}P(|\frac{1}{\sqrt{n}}y(t-j)M_t\varepsilon(t)|>\lambda)$$

$$<\frac{\delta^4}{\lambda^4n^2}\sum Ey^4(t-j)\beta_t^4<\frac{\delta^4M\omega}{\lambda^4n}\longrightarrow 0,\ \text{as}\ n\to\infty$$

Hence

$$r_n^j\xrightarrow{P}0,\ \text{as}\ n\to\infty\ \ j=1,2,\cdots,n_a \tag{2.7}$$

From (2.6) and (2.7), we obtain that

$$\max_{t\le n}|\frac{1}{\sqrt{n}}\phi(t)M_t\varepsilon(t)|\xrightarrow{P}0,\ \text{as}\ n\to\infty$$

The proof is complete.

Lemma 5. All the conditions of Lemma 4 are satisfied, then

$$E(\|\max_{t\le n}|\frac{1}{\sqrt{n}}\phi(t)M_t\varepsilon(t)|\|)^2<T<\infty \tag{2.8}$$

where $\|\cdot\|$ denotes Euclid norm and T is a constant.

Proof. Since

$$E(w_n^l)^2 \leq \frac{1}{n}\sum_{t=1}^{n} E^{n^2}u(t-l)M_t^2\varepsilon^2(t) \leq \delta^2\sqrt{r}W = T_1 < \infty$$

$$\forall l = 1,2,\cdots,n_b$$

and

$$E(r_n^j)^2 \leq \frac{\delta^2}{n}\sum_{t=1}^{n} Er^2(t-j)\Lambda_t^2 \qquad j = 1,2,\cdots,n_a$$

$$\leq \delta^2\sqrt{MW} = T_2 < \infty$$

$$T = \max\{T_1, T_2\}$$

then we obtain (2.8).

Lemma 6. If C1, C2, C3 and C4 are satisfied and

$$M_t = M \quad , \quad \Lambda_t^2 = \Lambda^2$$

then

$$\frac{1}{n}\sum_{t=1}^{n} M^2\phi(t)\phi^T(t)\varepsilon^2(t) \longrightarrow R, \quad \text{as } n\to\infty \tag{2.9a}$$

$$P_n \longrightarrow P, \qquad \text{as } n\to\infty$$

provided that (t) (t) and (t) (t) (t) are all ergodic.

Moreover

$$R = M^2\Lambda^2 E\phi\phi^T \tag{2.9b}$$

$$P = (ME\phi\phi^T)^{-1} \tag{2.9c}$$

Proof. With the ergodicity of $\phi(t)\phi^T(t)$ and $\varepsilon^2(t)\phi(t)\phi^T(t)$ we obtain that

$$\frac{1}{n}\sum_{t=1}^{n} M^2\varepsilon^2(t)\phi(t)\phi^T(t)$$

$$= M^2\cdot\frac{1}{n}\sum_{t=1}^{n}\varepsilon^2(t)\phi(t)\phi^T(t) \longrightarrow M^2\Lambda^2E\phi\phi^T$$

$$P_n = \left[\frac{1}{n}\sum_{t=1}^{n} M\phi(t)\phi^T(t)\right]^{-1} \longrightarrow M^{-1}(E\phi\phi^T)^{-1}$$

Lemma 7 (Mcleish's Theorem, [2]) Let $X_{n,k}$ be a martingale difference array satisfying

(a) $\max\limits_{k\leq k_n}|X_{n,k}|$ is uniformly bounded in L_2 norm, i.e., (2.8),

(b) $\max\limits_{k\leq k_n}|X_{n,k}| \xrightarrow{P} 0$, i.e., (2.5) and

(c) $\sum\limits_{k=1}^{k_n} X_{n,k}^2 \xrightarrow{P} R$, i.e., (2.9)

then AsN$(0, R)$.

Obviously,
$$\sqrt{n}(\hat{\theta}_n - \theta) = P_n\cdot\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\phi(t)\varepsilon(t)$$

Using Mcleish's Theorem (Lemma 7) and Lemma 4—Lemma 6, we obtain the following theorem.

Theorem 1. If the conditions of Lemma 4, Lemma 5 and Lemma 6 are satisfied then

$$\sqrt{n}(\hat{\theta}_n-\theta)\in AsN(0,Q) \tag{2.10}$$
$$Q=\Lambda^2(E\phi\phi^T)^{-1}$$

Some Conclutions

Theorem 1 is more general than that of Ljung's, in the sense where $\varepsilon(t)$ is a white noise and $E\varepsilon^4(t)$ is bounded [3]. Of course this theorem has improved the results in[1].

All the assertions mentioned above can also be generalized to the MIMO situation.

References

[1] Yuan Zhen Dong (1982), Journal of Mathematical Research and Exposition, Vol.2 No.3, 71—78.

[2] D. L. Mcleish (1974), The Anuals of Prob. Vol.2 No.4, pp. 620—628.

[3] L. Ljung and Soderstrom (1983), Theory and Practice of Recursive Identification, MIT Press .

系统辨识中LS估计的渐近正态性的注记

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摘　要

文献[1]中，我们用有关鞅的中心极限定理，证明了系统辨识中LS估计的渐近正态性．然而[1]中的条件是苛刻的．本文利用Mcleish的相依变量的中心极限定理改进了[1]的结果．