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A Kind of Partition Identity*

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Let $\sigma(n)$ denote the set of partitions of n, usually denoted by $1^{k_1}2^{k_2}\cdots n^{k_n}$ with $k_1+2k_2+\cdots+nk_n=n$. Let $\phi_i(t)$ $(i=1,\cdots,r)$ be power series in t (over complex field $\mathbf{C}$) with positive radii of convergence and $\phi_i(0)=1$. For every $\bar{a}\equiv(a_1,\cdots,a_r)\in\mathbf{C}^r$ let the power-type generating function

$$G(t)\equiv G(t;\bar{a})=\prod_{i=1}^{r}(\phi_i(t))^{a_i} \qquad (1)$$

have a power series expansion in t which we may denote by

$$G(t)=\exp(\log G(t))=\sum_{n\geq 0}\left\{{\bar{a} \atop n}\right\}t^n,\quad \left\{{\bar{a} \atop 0}\right\}=G(0)=1. \qquad (2)$$

Theorem With (1) and (2) as given above we have

$$\left\{{\lambda\bar{a} \atop n}\right\}=\sum_{\sigma(n)}\binom{\lambda}{k}\prod_{i=1}^{n}\left\{{\bar{a} \atop i}\right\}^{k_i}, \qquad (3)$$

where $\lambda\in\mathbf{C}$, $\lambda\neq 0$, $\lambda\bar{a}\equiv(\lambda a_1,\cdots,\lambda a_r)$, and the summation extends over all partitions of n, and $\binom{\lambda}{k}$ represents the multinomial coefficient with $\bar{k}:=(k_1,\cdots,k_n)$, viz. $\binom{a}{k}=(\lambda)_k/\pi k_i!$, $k=k_1+\cdots+k_n$.

This theorem implies various special partition identities, e.g. those involving binomial coefficients due to W. C. Chu, Pascal-T triangle numbers of higher orders, generalized Bernoulli/Euler numbers, two kinds of Stirling numbers, Gegenbauer and Humbert polynomials, etc. respectively. In particular, (3) may also be used to yield a formula for Jacobi's hypergeometric polynomials $F(-n,b,c,z)$ with real $\lambda\neq 0$, viz.

$$\binom{-\lambda c}{n}F(-n,\lambda b,\lambda c,z)=\sum_{\sigma(n)}\binom{\lambda}{k}\prod_{i=1}^{n}\left(\binom{-c}{i}F(-i,b,c,z)\right)^{k_i}. \qquad (4)$$

All details will appear elsewhere.

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