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An Application of Necessary and Sufficient Condition of $\mathscr{C}$-Equivalence *

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In singularity theory and catastrophe theory of germs of C^∞ functions, many fundamental problems deal with computation of a basis of an arbitrary complementary space of an ideal of finite condimension in E_n, which is a ring consisting of all germs of C^∞ functions in n variables. For example, J.N. Mather has proved the following fundamental theorem with respect to universal deformation of a germ of finite codimension: Let $f \in E_n$ be a germ of finite codmension. Then a p-parameter deformation F of f is universal, if and only if its initial speeds are such that

$$J(f) + R\{\dot{F}_1, \dot{F}_2, \cdots, \dot{F}_p\} = E_n .$$

This means that if a p-parameter universal deformation of f is to be found out, we need only to find out a basis of complementary space of $J(f)$ in E_n. Where $J(f)$ denotes the Jacobian ideal of f.

However, in the course of practical computation, it often makes troubles. But in the published literatures we seldom see something concerning with it and introducing it. In [3](p.13-p.15), several examples have been given and made computation by a diagram. But we must point out that (i) The generators in the enumerated examples in the paper are all simplest-monomials. The concerned variables are only two. (ii) In the paper, the general principle that computes the problem is not concerned, and such diagram can not be drawn when the number of variables $\geqslant 3$. For the more complicated generators, for instance polynomials, the diagram is no longer suitable. Starting from a proposition of $\mathscr{C}$-equivalence which has been given by J.N. Mather in [2], this article will use some algebraic knowledge to draw some theorems and principles of computing the problem. Finally, it will also give some pratical examples for explaination.

§ I. Preliminaries and symbols

⟨1⟩ The set of all homogeneous polynomials of degree K in n variables is

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denoted by P_n^K. Obviously, this is a vector space over R. In P_n^K, all different monomials with coefficient 1 form a basis in the space. Henceforth, we shall call canonical basis in this paper.

〈2〉 Let I be an ideal in E_n. The unique maximal ideal of E_n is denoted by $M \subset E_n$. $I_K = I/M^{K+1} = J^K(I) \subset E_n/M^{K+1}$ denotes the projection of I into J_n^K. Then $M^K \subset I$ is equivalent to $I_K \supset M^K/M^{K+1} = P_n^K$.

〈3〉 Assume that I is an ideal in E_n. then the following propositions are equivalent:

a) I is finite codimensional in E_n as a vector space over R.

b) There exists a non-negative integer K such that $I \supset M^K$.

In this paper, except those specially stipulated, the symbols are the same as those in [1].

§2. The main results

J.N.Mather has proved the following proposition in [2]:

If $f, g \in F$, then the following conditions are equivalent:

a) f is in the same $\mathscr{C}$-orbit as g.

b) $I(f) = I(g)$.

c) There exists an invertible $p \times p$ matrix (u_{ij}) such that $f^*(y_i) = \sum_j u_{ij} g^*(y_j)$. Where, $u_{ij} \in C(N)_s$.

d) There exists an invertible C^∞ map-germ $H|(N \times P, S \times y) =$ identity and H (graph f) = graph g.

Where, N and P are differentiable manifolds, S is a finite subset of N, $y \in P$, F denotes the set of all C^∞ map-germs $f:(N,S) \to (P, y)$, From this, we can directly conclude that

Lemma 1. Assume that $I^{(1)}$ and $I^{(2)}$ are two finitely generated ideals in E_n: $I^{(1)} = [f_1, f_2, \cdots, f_m]$, $I^{(2)} = [g_1, g_2, \cdots, g_m]$, then $I^{(1)} = I^{(2)}$ if and only if there exists an invertible matrix $U = (u_{ij})$ in E_n ($u_{ij} \in E_n$, $i, j = 1, 2, \cdots, m$) such that

$$\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix} = (u_{ij}) \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{bmatrix}$$

Theorem 1. Assume that I is a finitely generated ideal in E_n: $I = [f_1, f_2, \cdots, f_m]$, and it has finite codimension: $I \supset M^K$ (suppose that K is smallest natural integer satisfying the relation). Then $[f_1, f_2, \cdots, f_m] = [j^K f_1, j^K f_2, \cdots, j^K f_m]$. Where, $j^K f_i$ is the Taylor polynomial of f_i up to degree K.

Proof. Assume that $f_i = j^K f_i + g_i$, then $g_i \in M^{K+1}$ Moreover, $I \supset M^K$. Therefore, $M \cdot I \supset M^{K+1}$.

From this, $g_i=\sum_{j=1}^{m} c_{ij} f_j$, Where, $c_{ij}\in M$. Therefore, $f_i=j^K f_i+\sum_{j=1}^{m} c_{ij} f_j$ i.e.,

$$[E-(c_{ij})]\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}=\begin{bmatrix} j^K f_1 \\ j^K f_2 \\ \vdots \\ j^K f_m \end{bmatrix}$$

(Where, E is an $n\times n$ identity matrix).

Note that $[E-(c_{ij})]$ is an invertible matrix in E_n. By Lemmal, we have

$$[f_1, f_2, \cdots, f_m]=[j^K f_1, j^K f_2, \cdots, j^K f_m].$$

The theorem points out that any finitely generated and finite codimensional ideal in E_n can be replaced by an ideal whose generators are all polynomials for the relavent computations.

Theorem 2. Assume that I is a finitely generated ideal: $I=[f_1, f_2, \cdots, f_m]$.

(i) Write the set consisting of al l invertible germs in E_n as V_n.

If $h_j\in V_n (j=1,2,\cdots,m)$, then $[f_1, f_2, \cdots, f_m]=[f_1, \cdots, h_i f_i, \cdots, f_m]=[h_1 f_1, \cdots, h_i f_i, \cdots, h_m f_m]$.

(ii) If $h\in E_n$, then $[f_1, f_2, \cdots, f_m]=[f_1, \cdots, (hf_j+f_i), \cdots, f_m]$. $(j\neq i)$

(iii) If $h\in E_n$ and $h(0)\neq -1$, then $[f_1, f_2, \cdots, f_m]=[f_1, \cdots, (hf_i+f_i), \cdots, f_m]$.

Proof. (i) If $h_j\in V_n$, then the matries

$$H_1=\begin{bmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & & & h_i & & \\ & 0 & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \quad\text{and}\quad H_2=\begin{bmatrix} h_1 & & & & \\ & \ddots & & 0 & \\ & & h_i & & \\ & 0 & & \ddots & \\ & & & & h_m \end{bmatrix}$$

are invertible in E_n. But, $[f_1, \cdots, f_i, \cdots, f_m]=[f_1, \cdots, h_i f_i, \cdots, f_m]$ is equivalent to

$$\begin{bmatrix} f_1 \\ \vdots \\ h_i f_i \\ \vdots \\ f_m \end{bmatrix}=H_1\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}$$

$[f_1, \cdots, f_i, \cdots, f_m]=[h_1 f_1, \cdots, h_i f_i, \cdots, h_m f_m]$

is equipalent to

$$\begin{bmatrix} h_1 f_1 \\ h_2 f_2 \\ \vdots \\ h_m f_m \end{bmatrix}=H_2\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}$$

By lemmal, we know that (i) is true. (ii) and (iii): Similarly to (i), we need only to notece that the following matries

$$\begin{array}{cl}\begin{bmatrix} 1 & & & & & & \\ \vdots & \ddots & & & & 0 & \\ 1 & \cdots & 1 & & & & \\ \vdots & & \vdots & \ddots & & & \\ 0 & \cdots & h & \cdots & 1 & & \\ \vdots & & \vdots & & \vdots & \ddots & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} & i\,\text{row} \\ \quad\; j\text{column} & \end{array} \qquad \begin{array}{cl}\begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ & \ddots & \vdots & & \vdots & & \vdots \\ & & 1 & \cdots & h & \cdots & 0 \\ & & & \ddots & \vdots & & \vdots \\ & 0 & & & 1 & \cdots & 0 \\ & & & & & \ddots & \vdots \\ & & & & & & 1 \end{bmatrix} & i\,\text{row}\,(h\in E_n) \\ \quad\; j\ \text{column} & \end{array}$$

and

$$\begin{bmatrix} 1 & & & \\ & \ddots & & 0 \\ 0 & & 1+h & \\ & & & \ddots \\ & & & & 1 \end{bmatrix} \quad (h\in E_n,\ h(0)\neq -1)$$

are all invertible in E_n.

Henceforth, the above operations will be called"elementary transformations" of a finitely generated ideal in E_n. When the operations are used, special attention should be paid that the designative ranges of h. Can't be used at random.

Corollary 1. Assume that $I=[f_1,f_2,\cdots,f_m]$, then the following conditions are equivalent:

(i) $g\in[f_1,f_2,\cdots,f_m]$.

(ii) $[f_1,f_2,\cdots,f_m]$ can be obtained from $[f_1,f_2,\cdots,f_m,g]$ by performing a series of the "elementary transformations".

Corollary 2. Assume that $I=[f_1,f_2,\cdots,f_m]$, $g\in M\cdot I$, then $[f_1,f_2,\cdots,f_m]=[f_1\cdots,f_i+g,\cdots,f_m]$.

In fact, if $g\in M\cdot I$, i.e. $g=\sum_{i=1}^{m}h_if_i$, $h_i\in M$, Hence, I can be obtained from $[f_1,\cdots,f_i+g,\cdots,f_m]$ by performing a series of the "elementary transformations".

The simple form of generators is very useful for the computations. However, theorem 2 and its corollaries are simple and important tools for the realization of the aim. Some examples can be seen in §3.

Assume that I is an ideal of finite codimension in E_n. Consider the sequence of nested ideals: $E_n\supset I+M\supset I+M^2\cdots\supset I+M^K$. Suppose the codimension of $(I+M^{r+1})$ in $(I+M^r)$ is C_r $(0<r<K-1)$, then the codimension of I in E_n is Equal to $C_0+C_1+\cdots+C_{K-1}$.

Lemma 2. C_r is the codimension of $(I_r\cap P_n^r)$ in P_n^r.

Proof. Since $C_r=\dim_R(I+M^r)/(I+M^{r+1})$. Moreover.

$$(I+M^r)/(I+M^{r+1})=(I+M^{r+1})+M^r/(I+M^{r+1})=M^r/(I+M^{r+1})\cap M^r$$
$$=(M^r/M^{r+1})/((I+M^{r+1})\cap M^r/M^{r+1})=j^r(M^r)/j^r((I+M^{r+1})\cap M^r)$$

$$= P_n^r / j^r\ ((I + M^{r+1}) \cap M^r) = P_n^r / (I_r \cap P_n^r).$$

Therefore, $\quad C_r = \dim_R (I + M^r)/(I + M^{r+1}) = \dim_R P_n^r / (I_r \cap P_n^r).$

Lemma 3. $I_r \cap P_n^r$ is a R-subspace in P_n^r.

Proof. It is clear.

Theorem 3. Suppose that I is an ideal of finite codimension in E_n, then E_n/I as a vector space over R has the following decomposition of the direct sum: $E_n/I = \bigoplus_{r=0}^{k-1} \overline{(I_r \cap P_n^r)}$. Where, $\overline{(I_r \cap P_n^r)}$ is a complementary subspace of $I_r \cap P_n^r$ in P_n^r.

Proof. By lemma 2, we know that the complementary sabspace of $I_r \cap P_n^r$ in P_n^r is just a complementary subspace of $(I + M^{r+1})$ in $(I + M^r)$.

Therefore, $(I + M^{r+1}) \oplus \overline{(I_r \cap P_n^r)} = (I + M^r)$.

From this, $\quad E_n = \overline{(I_0 \cap P_n^0)} \oplus (I + M) = \overline{(I_0 \cap P_n^0)} \oplus \overline{(I_1 \cap P_n^i)} \oplus (I + M^2)$

$$= \cdots = \bigoplus_{r=0}^{k-1} \overline{(I_r \cap P_n^r)} \oplus (I + M^K).$$

But $I + M^K = I$ (Because $I \supset M^K$).

Hence, $\quad E_n = \bigoplus_{r=0}^{k-1} \overline{(I_r \cap P_n^r)} \oplus I$, i.e. $\quad E_n/I = \bigoplus_{r=0}^{k-1} (\overline{(I_r \cap P_n^r)}$

It is known from the linear algebra that: The complementary subspace of $(I_r \cap P_n^r)$ in P_n^r is not unique. In same space, we may choose different basis too. Our problem is to find out a basis of an arbitrary complementary subspace of $I_r \cap P_n^r$ in P_n^r. Without loss of generality, suppose that $\overline{I_r \cap P_n^r}$ is such a complementary subspace of $I_r \cap P_n^r$ in P_n^r, it has a basis which al l consist of some elememts of the canonical basis in P_n^r. Thus, $\bigoplus_{r=0}^{k-1} \overline{(I_r \cap P_n^r)}$ will such a complementary subspace of finite codimension in E_n: It has a basis which al l co-sist of some elements of the canonical basis. Our aim is to find out the basis.

Theorcm 4. Assume that the above mentioned basis in $\overline{I_0 \cap P_n^0}$, $\overline{I_1 \cap P_n^1}, \cdots, \overline{I_r \cap P_n^r}$ have obtained, and its direct sum is written as $R\{g_1, g_2, \cdots, g_s\}$, then an element of the canonical basis in P_n^r (where $r < K-1$) is a basis element in $\overline{I_r \cap P_n^r}$, if and only if the element no belongs to $I \oplus R\{g_1, g_2, \cdots, g_s\}$.

Proof. (i) The necessity: By theorem 3, we have $E_n = \bigoplus_{r=0}^{k-1} \overline{(I_r \cap P_n^r)} \oplus I = I \oplus R\{g_1, g_2, \cdots, g_s\} \oplus \overline{(I_r \cap P_n^r)} \oplus \cdots \oplus \overline{(I_{K-1} \cap P_n^{K-1})}$.

If ξ is a canonical basis element and ξ is also a basis element of $\overline{(I_r \cap P_n^r)}$. Obviously, $\xi \bar{\in} I \oplus R\{g_1, g_2, \cdots g_s\}$.

(ii) The sufficiency: Indeed, $I \oplus R\{g_1, g_2, \cdots, g_s\} = P_n^0 \oplus P_n^1 \oplus \cdots \oplus P_n^{r-1} \oplus (I_r \cap P_n^r) \oplus \cdots \oplus (I_{K-1} \cap P_n^{K-1}) \oplus M^K$.

If ξ is a canonical basis element in P_n^r (i.e., ξ is a monomial of degree r with coefficient 1) and

$$\xi \overline{\in} I \oplus R\{g_1, g_2, \cdots, g_s\}.$$

Then it is easy to know that the above conditions are equivalent to that the projection of ξ into $I \oplus R\{g_1, g_2, \cdots, g_s\}$ is zero. Moreover, by $E_n = I \oplus R\{g_1, g_2, \cdots, g_s\} \oplus \overline{(I_r \cap P_n^r)} \oplus \cdots \oplus \overline{(I_{K-1} \cap P_n^{K-1})}$, we know that $\xi \in \overline{(I_r \cap P_n^r)} \oplus \overline{(I_{r+1} \cap P_n^{r+1})} \oplus \cdots \oplus \overline{(I_{K-1} \cap P_n^{K-1})}$. But, the projections of ξ into $\overline{(I_{r+1} \cap P_n^{r+1})}, \cdots \overline{(I_{K-1} \cap P_n^{K-1})}$ respectively are all zero. Therefore, $\xi \in \overline{(I_r \cap P_n^r)}$.

Because the considered basis elements of $\overline{(I_r \cap P_n^r)}$ are some elements of the canonical basis in P_n^r.

From this, ξ is certainly a basis element of $\overline{(I_r \cap P_n^r)}$.

Corollary 3. Arbitrary a canonical basis element in P_n^r belongs to $I \oplus R\{g_1, g_2, \cdots, g_s\}$.

Proof. By $I \oplus R\{g_1, g_2, \cdots, g_s\} = P_n^0 \oplus P_n^1 \oplus \cdots \oplus P_n^{r-1} \oplus (I_r \cap P_n^r) \oplus \cdots \oplus (I_{K-1} \cap P_n^{K-1}) \oplus M^K$, We may directly conclude the result.

When we find a basis of $\overline{(I_r \cap P_n^r)}$, $\overline{(I_r \cap P_n^r)}$ can be again regarded as a direct sum of all 1-dimensional subspaces generated by every basis element. Thus, if we have determined $\xi \overline{\in} R\{g_1, g_2, \cdots, g_s\}$, then ξ is found a basis element of $\overline{(I_r \cap P_n^r)}$, Next, we should add ξ to $R\{g_1, g_2, \cdots, g_s\}$ and obtain $I \oplus R\{g_1, g_2, \cdots, g_3, \xi\}$. Based on the result, over again consider such canonical basis element which no belongs to $I \oplus R\{g_1, g_2, \cdots, g_s, \xi\}$. In this manner, step by step, we may find out a basis of $\overline{(I_r \cap P_n^r)}$. Sometimes, there are some difficulties to directly use theorem 4. For the convenience of observation and computation, it is necessary to draw the following definitions and theorems.

Definition 1. Let $f \in E_n$. For any given integer $r \geqslant 0$, $j^{r-1} f$ will be called lower degree parts of the degree $< r$ of f. $(f - j^{r-1} f)$ will be called higher degree parts of the degree $\geqslant r$ of f and written as $H^r f$. (If $r = 0$, naturaly $j^{-1} f \overset{\text{def}}{=\!=} 0$).

Definition 2. Let ξ be a canonical basis element in P_n^r for given integer $r \geqslant 0$, $f \in E_n$. If there is a germ $\eta \in E_n$ such that the term of ξ appears in the Taylor expanxion of $f \cdot \eta$ (i.e., the coefficient of the term is not zero), then f will be called a germ possibly generating ξ.

Theorem 5. Let ξ is a canonical basis element in P_n^r, I is a finite generated and finite codimesional ideal: $I = [\psi_1, \psi_2, \cdots, \psi_q, \varphi_{q+1}, \cdots, \varphi_m]$. (Where, $\psi_1, \psi_2, \cdots, \psi_q$ are all of the germs possiby generating ξ in the generators of I), then $\xi \in I \oplus R\{g_1, g_2, \cdots, g_s\}$ if and only if there are polynomials $\eta_1, \eta_2, \cdots, \eta_q$ such that

(i) $H^r(\sum_{i=1}^{q} \eta_i j^{K-1} \psi_i)$ contains the term of ξ

(i.e., the coeffocient of the term is not zero)

(ii) $\{H^{r}(\sum_{i=1}^{q}\eta_i j^{K-1}\psi_i)$ − the term of $\xi\}\in I$.

Proof. The sufficiency: Since

$$\sum_{i=1}^{q}\eta_i\psi_i=\sum_{i=1}^{q}\eta_i(j^{K-1}\psi_i+H^K\psi_i)=\sum_{i=1}^{q}\eta_i j^{K-1}\psi_i+\sum_{i=1}^{q}\eta_i H^K\psi_i \ .$$

obviously

$$\sum_{i=1}^{q}\eta_i\psi_i\in I,\quad \sum_{i=1}^{q}\eta_i H^K\psi_i\in M^K\subset I .$$

Therefore

$$\sum_{i=1}^{q}\eta_i j^{K-1}\psi_i\in I .$$

Moreover

$$\sum_{i=1}^{q}\eta_i j^{K-1}\psi_i=j^{r-1}(\sum_{i=1}^{p}\eta_i j^{K-1}\psi_i)+H^r(\sum_{i=1}^{q}\eta_i j^{K-1}\psi_i) .$$

By corollary 3, we know that

$$j^{r-1}(\sum_{i=1}^{q}\eta_i j^{K-1}\psi_i)\in I\oplus R\{g_1,g_2,\cdots,g_s) .$$

Thereby

$$H^r(\sum_{i=1}^{q}\eta_i j^{K-1}\psi_i)\in I\oplus R\{g_1,g_2,\cdots,g_s)$$

Therefore, the term of $\xi=(\sum_{i=1}^{q}\eta_i j^{K-1}\psi_i)-j^{r-1}(\sum_{i=1}^{q}\eta_i j^{K-1}\psi_i)-$

$$-\{H^r(\sum_{i=1}^{q}\eta_i j^{K-1}\psi_i)-\text{the term of }\xi\}\in I\oplus R\{g_1,g_2,\cdots,g_s\} .$$

The necessity: Let ξ is a canonical basis element in P_n^r and $\xi\in I\oplus R\{g_1,g_2,\cdots,g_s\}$.

Note tnat the projection of ξ into $R\{g_1\ g_2,\cdots,g_s\}$ is zero, thereby $\xi\in I$.

From this $\xi=\sum_{i=1}^{q}h_i\psi_i+\sum_{j-1}^{m-q}h_{q+j}\varphi_{q+j}$. Therefore

$$\sum_{i=1}^{q}h_i\psi_i-\xi=-\sum_{j=1}^{m-q}h_{q+j}\varphi_{q+j}\in I .$$

Hence

$$H^r(j^{K-1}(\sum_{i=1}^{q}h_i\psi_i-\xi))=H^r(-j^{K-1}(\sum_{j=1}^{m-q}h_{q+j}\varphi_{q+j}))$$

$$=-\{\sum_{j=1}^{m-q}h_{q+1}\varphi_{q+j}-j^{r-1}(\sum_{j=1}^{m-q}h_{q+j}\varphi_{q+j})-H^K(\sum_{j=1}^{m-q}h_{q+j}\varphi_{q+j})\}\in I\oplus R\{g_1,g_2,\cdots,g_s\}$$

Note that the projection of $H^r(-j^{K-1}(\sum_{j=1}^{m-q}h_{q+j}\varphi_{q+j}))$ into $R\{g_1,g_2,\cdots,g_s\}$ is zero.

Therefore,

$$H'(-j^{K-1}(\sum_{j=1}^{m-q} h_{q+j}\varphi_{q+j}))\in I.$$

Thereby $$\{H'(j^{K-1}(\sum_{i=1}^{q} h_i\psi_i)) - \xi\}\in I.$$ But

$$H'(j^{K-1}(\sum_{i=1}^{q} h_i\psi_i)) = H'(j^{K-1}(\sum_{i=1}^{q} j^{K-1}h_i\cdot j^{K-1}\psi_i))$$

$$= H'[\sum_{i=1}^{q} j^{K-1}h_i\cdot j^{K-1}\psi_i - H^K(\sum_{i=1}^{q} j^{K-1}h_i\cdot j^{K-1}\psi_i)]$$

$$= H'(\sum_{i=1}^{q} j^{K-1}h_i\cdot j^{K-1}\psi_i) - H^K(\sum_{i=1}^{q} j^{K-1}h_i\cdot j^{K-1}\psi_i).$$

Obviously $$H^K(\sum_{i=1}^{q} j^{K-1}h_i\cdot j^{K-1}\psi_i)\in M^K\subset I.$$

Therefore, the relation $\{H'(j^{K-1}(\sum_{i=1}^{q} h_i\psi_i)) - \xi\}\in I$ is equivalent to $\{H'(\sum_{i=1}^{q} j^{K-1}h_i\cdot j^{K-1}\psi_i) - \xi\}\in I$. we need only take $\eta_i = j^{K-1}h_i$ $(i=1,2,\cdots,q)$.

From theorem 5 and the above proof, we can se e that: For any finitely generated and finite codimensional ideal I in E_n and any canonical basis element ξ in P_n^r, we want to determine $\xi\in I\oplus R\{g_1, g_2, \cdots, g_s\}$, that is equivalent to deter mine $\xi\in I$. If there are difficulties to directly determine $\xi\in I$, then instead the conditions (i) and (ii) of theorem 5 can be used. All oprations need only to be considered in the range of polynomials.

§3. The computing examples

Example 1. Assume that $f_1(x_1), f_2(x_2), \cdots, f_n(x_n)$ are polynomials in $x_1, x_2, \cdots, x_n$ respectively, $x_1^{K_1}, x_2^{K_2}, \cdots, x_n^{K_n}$ are terms of the lowest degree in $f_1(x_1), f_2(x_2), \cdots, f_n(x_n)$ respectively. Then $[f_1, f_2, \cdots, f_n] = [x_1^{K_1}, x_2^{K_2}, \cdots, x_n^{K_n}]$.

Indeed, $f_i(x_i) = x_i^{K_i}h_i(x_i)(i=1,2,\cdots,n)$, $h_i(x_i)$ is an element of V_n. By (i) in theorem 2, we can obtain the result.

Example 2. Assume that every generator $f_i = \sum_{j=1}^{n} h_{ij}g_j$ in $[f_1, f_2, \cdots, f_n]$. for all $i=1,2,\cdots,n$, $h_{ii}\in V_n$. When $j>i$ (or $j<i$), $h_{ij}\in M$. Then $[f_1, f_2, \cdots, f_n] = [g_1, g_2, \cdots, g_n]$

Obviously, this can be concluded directly from lemma 1. Because in this case, $(h_{ij}(0))$ is an upper triangular matrix or a lower triangular. All element on the principal diagonal are not zero. Hence, the matrix (h_{ij}) is an invertible matrix in E_n.

In particularly, if every f_i is a polynomial as the from $x_1^{m_1^{(i)}} + x_2^{m_2^{(i)}} + \cdots + x_n^{m_n^{(i)}}$

and

$$m_1^{(1)}<\min\{m_1^{(2)},\cdots,m_1^{(n)}\},\ m_2^{(2)}<\min\{m_2^{(1)},m_2^{(3)},\cdots,m_2^{(n)}\}\cdots,$$
$$m_i^{(i)}<\min\{m_i^{(1)},\cdots,m_i^{(i-1)},m_i^{(i+1)},\cdots,m_i^{(n)}\},\cdots,\quad m_n^{(n)}<\min\{m_n^{(1)},m_n^{(2)},\cdots,m_n^{(n-1)}\},$$

then

$$[f_1,f_2,\cdots,f_n]=[x_1^{m_1^{(1)}},x_2^{m_2^{(2)}},\cdots,x_n^{m_n^{(n)}}].$$

For example $[x^4+y^{11}+z^{12},x^5+y^3+z^9,x^7+y^8+z^7]=[x^4,y^3,z^7]$.

In the above every f_i, if it absents some terms, but what it absents is not $x_i^{m_i^{(i)}}$, In this case,

we still have: $[f_1,f_2,\cdots,f_n]=[x_1^{m_1^{(1)}},x_2^{m_2^{(2)}},\cdots,x_n^{m_n^{(n)}}]$.

For example $[x^2+y^3,y^2+z^3,z^2+x^3]=[x^2,y^2,z^2]$.

Example 3. f: $(R^2,0)\to(R,0)$

$$(x,y)\mapsto x^3y^2+\sin xy.$$

Finding an universal deformation of f.

Solution: Since $\frac{\partial f}{\partial x}=3x^2y^2+y\cos xy$, $\frac{\partial f}{\partial y}=2x^3y+x\cos xy$. Therefore

$$J(f)=[\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}]=[3x^2y^2+y\cos xy,\ 2x^3y+x\cos xy]$$
$$=[y(\cos xy+3x^2y),\ x(\cos xy+2x^2y)].$$

Because $(\cos xy+3x^2y)$ and $(\cos xy+2x^2y)$ are invetrible in E_2, by (i) of theorem 2, obtain:

$$J(f)=[x,y]$$

From this, $$J(f)+R\{1\}=E_2\ .$$

Hence, 1-parameter universal deformation of f is $F(x,y,t)=x^3y+\sin xy+t$.

Example 4. Assume that $[f_1,f_2]=[xy+x^4y+y^6,\ y^2+x^3+xy^3]$ Finding a basis of complementary space of $[f_1,f_2]$ in E_2.

Solution. $[f_1,f_2]=[f_1-y^4f_2,f_2]=[xy+x^4y-x^3y^4-xy^7,\ y^2+x^3+xy^3]$
$=[xy(1-x^3-x^2y^3-y^6),\ y^2+x^3+xy^3]=[xy,\ y^2+x^3+xy^3]$
$=[xy,\ y^2+x^3]$.

We now compute a basis of complementary space of I in E_2 (i.e., E_{xy}):

It is easy to know that $(\overline{I_0\cap P_2^0})\oplus(\overline{I_1\cap P_2^1})=R\{1,x,y\}$.

Consider the canonial basis in P_2^2: x^2,xy,y^2.

Obviously, $x^2\notin I\oplus R\{1,x,y\}$. by theorem 4, x^2 should be a canonical basis element of $(\overline{I_2\cap P_2^2})$. Thus, we obtain $I\oplus R\{1,x,y,x^2\}$. But, $xy\in I\subset I\oplus R\{1,x,y,x^2\}$. Hence, xy is not a basis element of $(\overline{I_2\cap P_2^2})$.

The term y^2: In the generators of I, (y^2+x^3) is the only germ that possibly generates y^2.

By theorem 5, It depends on whether x^3 belongs to I as to whether y^2 belong to $I\oplus R\{1,x,y,x^2\}$ or not. But $x^3\notin I$. In fact, if $x^3\in I$, then there are polynomials $\eta_1(x, y)$, $\eta_2(x, y)$ such that $x^3 \equiv \eta_1(x, y)(y^2+x^3)+\eta_2(x, y)xy$. Set $x=0$, we have $\eta_1(0, y)y^2 \equiv 0$. Hence $\eta_1(x, y)=x \cdot \eta_1^1(x, y)$. Therefore $x^3 \equiv x\eta_1^1(x, y)(y^2+x^3)+\eta_2(x, y)xy$. Namely $x^2 \equiv \eta_1^1(x, y)(y^2+x^3)+\eta_2(x, y)y$. Hence $x^2 \in [y^2+x^3, y]=[x^3, y]$. That is impossible. From this, y^2 should be a basis element of $(\overline{I_2 \cap P_2^2})$. Thus we obtain $I \oplus R\{1, x, y, x^2, y^2\}$.

Seocndly, consider the basis of $(\overline{I_3 \cap P_2^3})$: In the generators of I. (y^2+x^3) is the unique generator possibly generating x^3. By corollary 3, y^2 can be not considered, because $y^2 \in R\{1, x, y, x^2, y^2\}$ But, in (y^2+x^3), the parts of the degree $\geqslant 3$ do not contain x^3 and ane $0 \in I$. From this, we know that $x^3 \in I \oplus R\{1, x, y, x^2, y^2\}$.

Again

$$x^2y \in I \subset I \oplus R\{1, x, y, x^2, y^2\},$$
$$xy^2 \in I \subset I \oplus R\{1, x, y, x^2, y^2\}.$$

Finally, consider y^3: In the generators of I. (y^2+x^3) is the only generator possibly generating y^3. However $y(y^2+x^3)=y^3+xy$ and includes y^3, also $x^3y=x^2(xy) \in I$. Therefore $y^3 \in I \oplus R\{1, x, y, x^2, y^2\}$.

It is easy to know that $I \supset M^4$. Therefore,

$$I \oplus R\{1, x, y, x^2, y^2\}=E_{x, y}=E_2.$$

Example 5. Assume that $I=[xy+y^6, x^3+xy^3]$, Finding a basis of comple mentary space of I in E_2.

I is finite codimensional in E_2: It is only necessary to show that there are natural integers m_1 and m_2 such that $x^{m_1} \in I$ and $y^{m_2} \in I$. In fact, $x^4 \in I$, $y^9 \in I$. Be-cause $x(x^3+xy^3)$ contains x^4, Hence $x^4 \in I$ if and only if $x^2y^3 \in I$.

Assume that $x^2y^3 \equiv c_1(x, y)(xy+y^6)+c_2(x, y)(x^3+xy^3)$.

Set $x=0$, we obtain $c_1(0, y)y^6 \equiv 0$. Therefore $c_1(0, y) \equiv 0$. From this, $c_1(x, y)=xc_1^1(x, y)$.

Similarly, set $y=0$, we have $c_2(x, y)=yc_\lambda^1(x, y)$. Hence,

$$x^2y^3=c_1^1(x, y)(x+y^5)xy+c_2^1(x^2+y^3)xy,$$

Namely

$$xy^2=c_1^1(x, y)(x+y^5)+c_2^1(x, y)(x^2+y^3).$$

This implies that

$$xy^2 \in [x+y^5, x^2+y^3].$$

But

$$[x+y^5, x^2+y^3]=[x+y^5, y^3-xy^5]=[x+y^5, y^3(1-xy^2)]$$
$$=[x+y^5, y^3]=[x, y^3].$$

Obviously, $xy^2 \in [x, y^3]$ By means of the result, we have $x^4 \in I$.

Similarly we may prove that $y^9 \in I$.

In the following, we shal l compute a basis of complementary space of I in

E_2:

Obviously,

$$\overline{(I_0 \cap P_2^0)} \oplus \overline{(I_1 \cap P_2^1)} \oplus \overline{(I_2 \cap P_2^2)} = R\{1, x, y, x^2, xy, y^2\}.$$

Examine every canonical basis element in P_2^3 one by one: Because $(x^3 + xy^3)$ is a generator of I, hence, it depends on whether xy^3 belongs to I as to whether x^3 belongs to $I \oplus R\{1, x, y, x^2, xy, y^2\}$ or not.

If $xy^3 \in I$, then

$$xy^3 \equiv c_1(x, y)(x + y^5)y + c_2(x, y)(x^2 + y^3)x.$$

Set $x = 0$, we conclude that $c_1(x, y) = xc_1'(x, y)$

$y = 0$, then $c_2(x, y) = yc_1'(x, y)$.

Therefore

$$xy^3 \equiv c_1'(x, y)xy(x + y^5) + c_2'(x, y)xy(x^2 + y^3),$$
$$y^2 = c_1'(x, y)(x + y^5) + c_2'(x, y)(x^2 + y^3)$$

i.e.,

$$y^2 \in [x + y^5, x^2 + y^3] = [x, y^3].$$

That is impossible. Hence,

$$x^3 \notin I \oplus R\{1, x, y, x^2, xy, y^2\}.$$

Adding x^3 to $R\{1, x, y, x^2, xy, y^2\}$, we obtain

$$I \oplus R\{1, x, y, x^2, xy, y^2, x^3\}.$$

Similarly

$$x^2y \in I \subset I \oplus R\{1, x, y, x^2, xy, y^2, x^3\},$$
$$xy^2 \bar{\in} I \oplus R\{1, x, y, x^2, xy, y^2, x^3\}.$$

Again adding xy^2 to $I \oplus R\{1, x, y, x^2, xy, y^2, x^3\}$, we obtain

$$I \oplus R\{1, x, y, x^2, xy, y^2, x^3, xy^2\}.$$

But

$$y^3 \bar{\in} I \oplus R\{1, x, y, x^2, xy, y^2, x^3, xy^2\}, \text{ Hence,}$$
$$I \oplus \overline{(I_0 \cap P_2^0)} \oplus \overline{(I_1 \cap P_2^1)} \oplus \overline{(I_2 \cap P_2^2)} \oplus \overline{(I_3 \cap P_2^3)}$$
$$= I \oplus R\{1, x, y, x^2, xy, y^2, x^3, xy^2, y^3\}.$$

Continuing to compute in the same method, we abtain

$$I \oplus R\{1, x, y, x^2, xy, y^2, x^3, xy^2, y^3, xy^3, y^4, y^5\} = E_{x,y} = E_2.$$

Example 6. Assume that

$$I = [x^3 + xyz^2 + y^3z^2 + z^6,\ y^2 + zy^2 + x^4y,\ xyz + xy^3].$$

In the process of performing "elementary transformations", we appoint that the first, second, third generator of the ideal denotes by (1), (2), (3) respectively. Thus

$$I \xlongequal{(2) - xyx(1)} [x^3 + xyz^2 + y^3z^2 + z^6,\ y^2 + zy^2 - x^2y^2z^2 - xy^4z^2 - xyz^6,\ xyz + xy^3]$$
$$\xlongequal{z^5 \times (3) + (2)} [x^3 + xyz^2 + y^3z^2 + z^6,\ y^2 + zy^2 - x^2y^2z^2 - xy^4z^2 + xy^3z^5,\ xyz + xy^3]$$
$$= [x^3 + xyz^2 + y^3z^2 + z^6,\ y^2(1 + z - x^2z^2 - xy^2z^2 + xyz^5),\ xyz + xy^3]$$
$$= [x^3 + xyz^2 + y^3z^2 + z^6,\ y^2,\ xyz + xy^3]$$
$$= [x^3 + xyz^2 + z^6, y^2, xyz] = [x^3 + z^6, y^2, xyz].$$

It is easy to show that $x^m \bar{\in} I$ for all natural integers m. From this, we can further conclude that I is infinite codimensional in $E_{x,y,z}$ (i.e. E_3).

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$\mathscr{C}$-等价充要条件的一个应用

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摘 要

在C^∞ 函数芽的奇点理论和突变论中，很多基本问题涉及到n个变元的函数芽环E_n中有限余维理想任一补空间一组基的计算．例如J.N.Maffler 对有限余维的函数芽的universal deformation 证明了下述基本定理：f的一个p——参数的deformation 是universal，当且仅当它的初速度$\dot{F}_i\,(i=1,2,\cdots,p)$使得：

$$J(f)+R\{\dot{F}_1,\dot{F}_2,\cdots,\dot{F}_p\}=E_n\ .$$

这意味着，若要把f的一个p—参数的universal deformation 求出来，只需把f的雅可比理想$J(f)$在E_n中一个补空间的一组基求出来。

但是，实际计算往往造成困难．而已出版的文献又很少介绍．文献[3](p.13—p.15)举了数例用了个图来进行计算．但必须指出：（i）该文举的数例，生成元都是最简单的情形—单项式，涉及的变元只是两个．（ii）文中并未给出计算这一问题的一般原则，且这样的图，当变元的个数 ⩾3时，已无法画出．对更复杂一点的生成元，例如多项式，也不再适用．本文将从$\mathscr{C}$-等价的充要条件的一个命题出发，利用某些代数知识，引出计算该问题的某些定理和原则．最后，举例加以说明．