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# On Global Asymptotic Stability of A Kind of Third-Order Nonlinear Systems*

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## I. Introduction and Preliminaries

It is well known that the global asymptotic stability of third-order ordinary differential systems has importance in applications. There are many results on this topic. See, for example [1]—[3]. Recently, the following more general third-order system

$$\dot{x}=y-h(x),\quad \dot{y}=\phi(z)-g(x),\quad \dot{z}=-f(x) \tag{1}$$

where

$$g\in C[R,R] \text{ and } h,\phi,f\in C^1[R,R] \tag{2}$$

was studied in [4]. The next theorem is the main result.

**Theorem A**([4]) Assume that (2) holds and that the following conditions are satisfied.

(i) $h(0)=0,\ \dfrac{f(x)}{x}>0$ for $x\neq 0$;

(ii) $\dfrac{\phi(z)}{z}>0$ for $z\neq 0,\ \int_0^{\pm\infty}\phi(z)ds=+\infty$;

(iii) there exists a positive number $B$ such that

$$B\frac{g(x)}{x}-\phi'(z)\geqq 0 \text{ and } h'(x)-Bf'(x)\geqq 0 \text{ for } x\neq 0$$

But not both of them equal to zero at the same point $x$.

Then the trivial solution of the system (1) is globally asymptoticly stable.

Since many third-order equations can be translated into the form of system (1), Theorem A has a extensive range of applications. By using Theorem A many results in [1]—[3] can be improved. See [4]. But, in many cases Theorem A also fails to apply. The following is a simple example.

**Example I** Consider the system

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$$\left.\begin{aligned} \dot{x} &= y - \left(\frac{1}{3}x^3 - x^2 + 2x\right) \\ \dot{y} &= z - ((x-1)^2 + 1)x \\ \dot{z} &= -x \end{aligned}\right\} \tag{3}$$

which is a special case of (1). It is not difficult to show that there does not exist a positive number $B$ such that the hypothesis (iii) of Theorem A is satisfied. However the trivial solution of (3) is globally asymptotically stable. See Corollary 1 below.

Our aim in this note is to relax the hypothesis (iii) of Theorem A. Our results extend and improve Theorem A and other results in [4].

**2. Improvement of Theorem A**

The main result in this section is the following theorem which is an improvement of Theorem A.

**Theorem 1** Assume that the hypotheses (i) and (ii) of Theorem A are satisfied. Suppose also that

(iii′) there exists a positive number $B$ such that

$$B\frac{g(x)}{x} - \phi'(z) \geqq 0 \text{ and } h'(x) - Bf'(x) \geqq 0 \text{ for } x \neq 0$$

but not both of them identically equal to zero in any interval of the variable $x$.

The the trivial solution of (1) is globally asymptotically stable.

**Sketch of the proof** First we will establish the following proposition.

**Proposition 1** Under the hypotheses of Theorem 1 the system (1) cannot have a solution $(x(t), y(t), z(t))$ with $x(t) \equiv C_0$ where $C_0$ is a nonzero constant.

Next, introduce the Liapunov function

$$V(x, y, z) = \int_0^x g(s)ds - x\phi(z) + B\int_0^z \phi(s)ds$$
$$+ \frac{1}{2}[y + Bf(x) - h(x)]^2 + \int_0^x [h'(s) - Bf'(s)]Bf(s)ds.$$

It is not difficult to prove that $V(x, y, z)$ is a definite positive function. Also

$$\dot{V}_{(1)}(x, y, z) = -[B\frac{g(x)}{x} - \phi'(z)]xf(x) - [h'(x) - Bf'(x)][y - h(x)]^2 \leqq 0$$

and so by using Proposition 1 one can show that the set

$$A = \{(x, y, z): \dot{V}_{(1)}(x, y, z) = 0\}$$

does not contain any positive half trajectory of the system (1) except for the trivial solution. Finally by an argument similar to that in [4] we can prove that all the positive half-trajectories are bounded. Therefore, the trivial solution of (1) is globally asymptotically stable.

The following result is an immediate consequence.

**Corollary 1** Assume that the hypotheses (i) and (ii) are satisfied. Suppose also that there exists a positive number $B$ such that

$$B\frac{g(x)}{x}-\phi'(z)\geqq 0 \text{ and } h'(x)-Bf'(x)\geqq 0 \text{ for } x\neq 0$$

but both of them simultaneously equal to zero at most at finite points in any finite interval of the variable $x$ Then the trivial solution of the system (1) is globally asymptoticly stable.

In Example 1 by taking $B=1$ it is easy to see that all the hypotheses of Corollary 1 are satisfied. Hence the trivial solution of (3) is globally asymptotically stable.

**3. Applications of Theorem 1**

In this section we will apply Theorem 1 to obtain stability results for some important third-order nonlinear equations.

First consider the equation

$$\dddot{x}+h(\dot{x})\ddot{x}+g(\dot{x})+\phi(x)=0 \tag{4}$$

where

$$h,g\in C[R,R] \text{ and } \phi\in C^1[R,R]. \tag{5}$$

Let $H(x)=\int_0^x h(s)ds$ and set

$$x_1=\dot{x},\ y_1=\ddot{x}+H(\dot{x}),\ z_1=-x.$$

By direct substitution Eq. (4) reduces to the system

$$\dot{x}_1=y_1-H(x_1),\ \dot{y}_1=-\phi(-z_1)-g(x_1),\ \dot{z}_1=-x_1.$$

Then by Theorem 1 we obtain the following result.

**Theorem 2** Assume that (5) holds,

$$\frac{\phi(z)}{z}>0 \text{ for } z\neq 0 \text{ and } \int_0^{\pm\infty}\phi(z)dz=+\infty.$$

Suppose also that there exists a positive number $B$ such that

$$B\frac{g(x)}{x}-\phi'(z)\geqq 0 \text{ and } h(x)\geqq B \text{ for } x\neq 0$$

but not both of them equalities simultaneously in any interval of the variable $x$. Then the trivial solution of Eq. (4) is globally asymptoticly stable with respect to $x$, $\dot{x}$ and $\ddot{x}$.

Next consider the equation

$$\dddot{x}+h(x)\ddot{x}+(g(x)+h'(x)\dot{x})\dot{x}+f(x)=0 \tag{6}$$

where

$$g\in C[R,R] \text{ and } f,h\in C^1[R,R]. \tag{7}$$

Let

$$H(x)=\int_0^x h(s)ds,\ G(x)=\int_0^x g(s)ds$$

and set

$$x=x,\quad y=\dot{x}+H(x)\quad \text{and}\quad z=\ddot{x}+h(x)\dot{x}+G(x).$$

Then by direct substitution Eq.(6) reduces to a system of the form of (1):

$$\dot{x}=y-H(x),\quad \dot{y}=z-G(x),\quad \dot{z}=-f(x).$$

Therefore, by Theorem 1 we have the following result.

**Theorem 3** Assume that there exists a positive number $B$ such that

$$\frac{f(x)}{x}>0,\quad \frac{G(x)}{x}\geqq B \quad \text{and}\quad Bh(x)\geqq f'(x)\quad \text{for}\quad x\neq 0$$

and that the last two inequlities do not become equalities simultaneously in any interval of the variable $x$. Then the trivial solution of Eq.(6) is globally asymptotically stable with respect to $\dot{x}$, $x$ and $\ddot{x}$.

## References


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# 关于一类三阶非线性系统的全局渐近稳定性


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## 摘 要

本文考虑三阶非线性系统

$$\dot{x}=y-h(x),\quad \dot{y}=\varphi(z)-g(x),\quad \dot{z}=-f(x) \qquad (1)$$

这里$g(x)$为连续函数，$h(x)$，$\varphi(z)$，$f(x)$为连续可微函数．我们改进文[4]中的结果，证明了如下的结论：

**定理** 若系统（1）满足如下条件：

(i) $h(0)=0$，当$x\neq 0$时，$\dfrac{f(x)}{x}>0$；

(ii) 当$z\neq 0$时，$\dfrac{\varphi(z)}{z}>0$，$\displaystyle\int_0^{\pm\infty}\varphi(z)\mathrm{d}z=+\infty$；

(iii) 存在常数$B>0$，使得当$x\neq 0$时，$B\dfrac{g(x)}{x}-\varphi'(z)\geqslant 0$，$h'(x)-Bf'(x)\geqslant 0$，且两式不同时在$x$的任何区间上恒为零，则其零解全局渐近稳定．

利用上述定理本文获得了几类重要的三阶非线性方程零解全局渐近稳定的充分条件，它们也是文[4]中相应结果的改进．