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Enumeration on the Labeled Graphs With K Cycles*

Ling Jie

(Guang Dong Institute of Technology)

Abstract

In this paper, we continue the study of the enumeration problems of labeled graphs with k cycles that Tomscu didn't solve. A counting formula for the connected graphs with n labeled vertices and k cycles intersecting at a point is given. As an application, together with Tomscu's result we solve the enumeration problem of labeled connected graphs with two cycles.

Harary raised a problem to enumerate labeled graphs with a given number of cycles in [1]. Renyi obtain a formula for counting the connected labeled graphs with order n and a cycle (see [2]). Tomscu gave a formula in [3] to enumerate the connected graphs which contain n labeled vertices and k cycles whose vertices are not intersected. In this paper we consider the case that k cycles are intersected. We obtain a counting formula for the connected graphs with n labeled vertices and k cycles intersecting at a point. As an application we give a counting formula for the connected labeled graphs with two cycles. we solve the enumeration problem of connected graphs with two cycles.

Lemma 1 (see [4])

The number of labeled trees with n vertices $x_1, x_2, \cdots x_n$ and $d(x_1)=r$ is

$$\binom{n-2}{r-1}\cdot(n-1)^{n-1-r}\quad .$$

Lemma 2 (see [5])

The number of Hamiltonian cycles in a complete graph k_n of order n is

$$(n-1)!/2$$

Theorem 1

The number of connected graphs with n labeled vertices and k cycles intersecting at a point is

$$\frac{(n-1)!}{2^k}\sum_{i=0}^{n-(2k+1)}\frac{(n-1)\cdot n^i}{i!}\sum_{(\lambda)}\frac{1}{\lambda_3!\lambda_4!\cdots\lambda_p!}$$

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where the second sum is over those partitions $(\lambda)=(\lambda_3,\lambda_4,\cdots,\lambda_p)$ of k such that

$$\begin{cases}\lambda_3+\lambda_4+\cdots+\lambda_p=k\\ 2\lambda_3+3\lambda_4+\cdots+(p-1)\lambda_p=n-i-1\end{cases}$$

Proof For each connected graph with exactly k cycles intersecting at a commom point, let k cycles be contracted to the commom vertex, then we obtain a tree; conversely, for each tree we use a k-cycle subgraph (which is composed of exactly k cycles intersecting at a commom point) instead of a point in the tree, then we obtain a connected graph which contains exactly k cycles intersecting at a commom point.

Suppose the specification of k cycles is $3^{\lambda_3}4^{\lambda_4}\cdots p^{\lambda_p}$, that is there are λ_3 cycles with length 3, λ_4 cycles with length $4,\cdots,\lambda_p$ cycles with length p. then the number of the vertices of k-cycle subgraph is $2\lambda_3+3\lambda_4+\cdots+(p-1)\lambda_p+1$

Let $$2\lambda_3+3\lambda_4+\cdots+(p-1)\lambda_p+1=n-i$$

1 Consider the labeled trees whose vertices are $x_0,x_1,\cdots,x_i$, $d(x_0)=r$.

By Lemma 1, the number of such trees is $\binom{i-1}{r-1}\cdot i^{i-r}$.

For each tree, we substitute a k-cycle subgraph for the vertex x_0, since there are r vertices of the tree join with x_0, there are $(n-1)^r$ different ways to join the r vertices of the tree to the $(n-i)$ vertices of k-cycle subgraph, and then we obtain a connected graph which contains n labeled vertices and exactly k cycles intersecting at a common point.

Hence, for each k-cycle subgraph, the number of labeled connected graphs which contain this k-cycle subgraph is

$$\sum_{r=0}^{i}\binom{i-1}{r-1}\cdot i^{i-r}(n-i)^r=(n-i)\cdot n^{i-1}.$$

2 Now we consider the number of k-cycle subgraphs of order $n-i$.

Since the common vertex of k-cycle subgraph can be any one of the $n-i$ vertices, we have $n-i$ different ways to choose the common point.

Separate the remaining $n-i-1$ vertices into λ_3 parts of two points, λ_4 parts of three points, $\cdots,\lambda_p$ parts of $p-1$ points, the number of ways to separate is

$$\binom{n-i-1}{\underbrace{2,\cdots,2}_{\lambda_3},\cdots,\underbrace{p-1,\cdots,p-1}_{\lambda_p}}=\frac{(n-i-1)!}{(2!)^{\lambda_3}(3!)^{\lambda_4}\cdots(p-1)!^{\lambda_p}}$$

Each part of $t-1$ points, together with the choosed common point can from a cycle with length t $(3\leqslant t\leqslant p)$. By Lemma 2, the number of such cycles is $(t-1)!/2$, thus, the number of k-cycle subgraphs is

$$(n-i)\cdot\frac{(n-i-1)!}{(2!)^{\lambda_3}\cdots(p-1)!^{\lambda_p}}\cdot\left(\frac{2!}{2}\right)^{\lambda_3}\cdots\left[\frac{(p-1)!}{2}\right]^{\lambda_p}\cdot\frac{1}{\lambda_3!}\cdots\frac{1}{\lambda_p!}=\frac{(n-i)!}{2^k\lambda_3!\,\lambda_4!\cdots\lambda_p!}$$

Note that the number of ways to choose the $n-i$ vertices of k-cycle subgraph is $\binom{n}{n-i}=\binom{n}{i}$, hence the number of connected graphs with n labeled points and k cycles of specification $3^{\lambda_3}4^{\lambda_4}\cdots p^{\lambda_p}$ intersecting at a common point is

$$\binom{n}{i}\cdot\frac{(n-i)!}{2^k\lambda_3!\cdots\lambda_p!}\cdot(n-i)\cdot n^{i-1}=\frac{(n-1)!}{2^k}\cdot\frac{(n-i)\cdot n^i}{i!}\cdot\frac{1}{\lambda_3!\cdots\lambda_p!}$$

Since the number of the vertices of k-cycle subgraphs satisfies:

$$2k+1\leq n-i\leq n$$

we have
$$0\leq i\leq n-2k-1.$$

For $i=0,1,2,\cdots,n-2k-1$, each partition $(\lambda_3,\cdots,\lambda_p)$ of k such that

$$\begin{cases}\lambda_3+\lambda_4+\cdots+\lambda_p=k\\ 2\lambda_3+3\lambda_4+\cdots+(p-1)\lambda_p=n-i-1\end{cases}$$

correspond to a kind of specification of k cycles $3^{\lambda_3}4^{\lambda_4}\cdots p^{\lambda_p}$ then the number of connected graphs with n labeled points and k cycles having a common point is

$$\sum_{i=0}^{n-2k-1}\sum_{(\lambda)}\frac{(n-1)!}{2^k}\cdot\frac{(n-1)n^i}{i!}\cdot\frac{1}{\lambda_3!\cdots\lambda_p!}$$

$$=\frac{(n-1)!}{2^k}\sum_{i=0}^{n-2k-1}\frac{(n-1)\cdot n^i}{i!}\sum_{(\lambda)}\frac{1}{\lambda_3!\cdots\lambda_p!}\qquad\blacksquare$$

where the second sum is over all the partitions $(\lambda)=(\lambda_3,\cdots,\lambda_p)$ of k such that

$$\begin{cases}\lambda_3+\lambda_4+\cdots+\lambda_p=k\\ 2\lambda_3+3\lambda_4+\cdots+(p-1)\lambda_p=n-i-1\end{cases}$$

For some special cases, we can derive more precise counting formulas. Now we consider the length of cycle as a parameter, we give following results.

Theorem 2

The number of connected graphs with n labeled points and k cycles of length s having a common point is

$$\frac{(n-1)!}{2^k\cdot k!}\cdot\frac{(k(s-1)+1)n^{n-k(s-1)-1}}{(n-k(s-1)-1)!}$$

Proof Since the length of each of the k cycles is s, the number of vertices of k-cycle subgraph is $k(s-1)+1$. Then the following equations

$$\begin{cases}\lambda_3+\lambda_4+\cdots+\lambda_p=k\\ 2\lambda_3+3\lambda_4+\cdots+(p-1)\lambda_p=n-i-1\end{cases}$$

has a unique solution

$$\begin{cases}\lambda_i=0 & i\neq s\\ \lambda_i=k & i=s\end{cases}$$

only if $i=n-k(s+1)-1$.

By theorem 1, the number of the required connected graphs is

$$\frac{(n-1)!}{2^k}\cdot\frac{(k(s-1)-1)\cdot n^{n-k(s-1)-1}}{(n-k(s-1)-1)!}\cdot\frac{1}{k!}\qquad\blacksquare$$

Theorem 3

The number of connected graphs which have n labeled points and contain

k cycles with lengths respectively $s_1, s_2, \cdots, s_k$ is

$$\frac{(n-1)!}{2^k} \cdot \frac{(\sum_{i=1}^{k} s_i - k + 1) \cdot n^{n+k-1-\sum_{i=1}^{k} s_i}}{(n-1+k-\sum_{i=1}^{k} s_i)!}$$

Proof For the k-cycle subgraph in which the lengths of the k cycles are respectively $s_1, s_2, \cdots, s_k$, the number of vertices is

$$(s_1-1)+(s_2-1)+\cdots+(s_k-1)+1=\sum_{i=1}^{k} s_i - k + 1$$

hence, only if $i = n - \sum_{j=1}^{k} s_j + k - 1$, the following equations

$$\begin{cases} \lambda_3+\lambda_4+\cdots+\lambda_p=k \\ 2\lambda_3+3\lambda_4+\cdots+(p-1)\lambda_p=n-i-1 \end{cases}$$

has a unique solution

$$\lambda_i=\begin{cases} 0 & i \neq s_j \\ 1 & i = s_j \end{cases}, \quad (1 \leqslant j \leqslant k).$$

By theorem 1, the number of the graphs required as in theorem 3 is

$$\frac{(n-1)!}{2^k} \cdot \frac{(\sum_{j=1}^{k} s_j - k + 1) \cdot n^{n-\sum_{j=1}^{k} s_j + k - 1}}{(n-\sum_{j=1}^{k} s_j + k + 1)!} \cdot \frac{1}{\lambda_{s_1}! \cdots \lambda_{s_k}!}$$

$$= \frac{(n-1)!}{2^k} \cdot \frac{(\sum_{j=1}^{k} s_j - k + 1) \cdot n^{n+k-1-\sum_{j=1}^{k} s_j}}{(n+k-1-\sum_{j=1}^{k} s_j)}$$

Corollary The number of connected graphs with n labeled vertices and two cycles of lengths respectively a, b is

$$\begin{cases} \frac{(a+b-1}{4}\binom{n}{a+b-1}(a+b-1)! \cdot n^{n-a-b}, & \text{if } a \neq b \\ \frac{2a-1}{8}\binom{n}{2a-1}(2a-1)! \cdot n^{n-2a}, & \text{if } a = b \end{cases}$$

Proof For $a=b$, take $s=a$, $k=2$ in theorem 2, we have immediately

$$\frac{(n-1)!}{2^2 \cdot 2!} \cdot \frac{(2(a-1)+1) \cdot n^{n-2(a-1)-1}}{(n-2(a-1)-1)!} = \frac{2a-1}{8}\binom{n}{2a-1}(2a-1)! \cdot n^{n-2a}$$

For the case $a \neq b$, take $s_1=a$, $s_2=b$ in theorem 3 we have

$$\frac{(n-1)!}{2^2} \cdot \frac{(a+b-2+1) \cdot n^{n-a-b+2-1}}{(n-a-b+2-1)!} = \frac{(a+b-1)(a+b-1)!}{4}\binom{n}{a+b-1} n^{n-a-b}.$$

Theorem 4

The number of connected graphs with n labeled points and two cycles is

$$\frac{(n-1)!}{4}\sum_{i=0}^{n-5}\frac{n^i}{i!}\Big(\sum_{(\lambda)}\frac{n-i}{\lambda_3!\cdots\lambda_p!}+\sum_{(\lambda')}\frac{n}{\lambda_3!\cdots\lambda_p!}\Big)$$

where the second sum is over those partitions $(\lambda)=(\lambda_3,\lambda_4,\cdots,\lambda_p)$ of 2 such that

$$\begin{cases}\lambda_3+\lambda_4+\cdots+\lambda_p=2\\ 2\lambda_3+\cdots+(p-1)\lambda_p=n-i-1\end{cases}$$

the third sum is over those partitions $(\lambda')=(\lambda_3,\lambda_4,\cdots,\lambda_p)$ of 2 such that

$$\begin{cases}\lambda_3+\lambda_4+\cdots+\lambda_p=2\\ 3\lambda_3+4\lambda_4+\cdots+p\lambda_p=n-i\end{cases}$$

Proof The position of two cycles in a graph may be in one of following cases: (1). the vertices of two cycles are not intersected; (2). the vertices of two cycles are intersected. We enumerate respectively

Case 1. Using the result of Tomescu in [3], we obtain the number of this kind of connected graphs:

$$\frac{n!}{4}\sum_{i=0}^{n-6}\frac{n^i}{i!}\sum_{(\lambda')}\frac{1}{\lambda_3!\,\lambda_4!\cdots\lambda_p!}\tag{1}$$

where the second sum is over those partitions $(\lambda')=(\lambda_3,\cdots,\lambda_p)$ of 2 such that

$$\begin{cases}\lambda_3+\lambda_4+\cdots+\lambda_p=2\\ 3\lambda_3+4\lambda_4+\cdots+p\lambda_p=n-i\end{cases}$$

Case 2. The vertices of two cycles are intersected. Then the two cycles can only intersect at a common vertex. By Theorem 1, the number of connected graphs with two cycles intersecting at a common point is

$$\frac{(n-1)!}{4}\sum_{i=0}^{n-5}\frac{(n-1)\cdot n^i}{i!}\sum_{(\lambda)}\frac{1}{\lambda_3!\cdots\lambda_p!}$$

where the second sum is over all the partitions $(\lambda)=(\lambda_3,\cdots,\lambda_p)$ of 2 such that

$$\begin{cases}\lambda_3+\lambda_4+\cdots+\lambda_p=2\\ 2\lambda_3+3\lambda_4+\cdots+(p-1)\lambda_p=n-i-1\end{cases}$$

Note that for $i=n-5$, the equations

$$\begin{cases}\lambda_3+\lambda_4+\cdots+\lambda_p=2\\ 3\lambda_3+4\lambda_4+\cdots+p\lambda_p=n-i\end{cases}$$

have no nonnegative integer solution. Thus, (1) can be expressed as

$$\frac{n!}{4}\sum_{i=0}^{n-5}\frac{n-i}{i!}\sum_{(\lambda')}\frac{1}{\lambda_3!\cdots\lambda_p!}$$

and then the number of connected graphs with n labeled vertices and exactly two cycles is

$$\frac{n!}{4}\sum_{i=0}^{n-5}\frac{n^i}{i!}\sum_{(\lambda')}\frac{1}{\lambda_3!\cdots\lambda_p!}+\frac{(n-1)!}{4}\sum_{i=0}^{n-5}\frac{(n-i)\cdot n^i}{i!}\sum_{(\lambda)}\frac{1}{\lambda_3!\cdots\lambda_p!}$$

$$=\frac{(n-1)!}{4}\sum_{i=0}^{n-5}\frac{n^i}{i!}\Big(\sum_{(\lambda')}\frac{n}{\lambda_3!\cdots\lambda_p!}+\sum_{(\lambda)}\frac{n-i}{\lambda_3!\cdots\lambda_p!}\Big)\qquad\blacksquare$$

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关于 K 圈标号图的计数

凌 捷

（广东工学院，广州）

本文研究含 K 个圈的标号图的计数问题，得出了有 n 个标定顶点且有 K 个交于一点的圈的连通图的计数公式，并得到了双圈连通标号图的计数公式，从而解决了 $K=2$ 时连通图的计数问题.