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Some Results on the Matrix Equation $A^m = \lambda J$*

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In this note, all matrices are of size $n\times n$ over integers. J denotes the matrix with all entries being 1. The matrix equation

$$A^m = \lambda J \tag{1}$$

where m, λ are integers, has been studied by several authors [1—7]. In this paper, we shall give a more general class of g-circulant solutions to (1).

Let $(a_0, a_1, \cdots, a_{n-1})$ be the first row of the g-circulant A.

$$Q_A(x) = \sum_{i=0}^{n-1} a_i x^i \tag{2}$$

is called the Hall Polynomial of A, sometimes write $Q(x) = Q_A(x)$ for short. Clearly, A is uniquely determined by $Q(x)$ and g. In this note A is a $(0,1)$ g-ciruulant, k is the sum of row entries, then (2) takes the form

$$Q(x) = \sum_{i=0}^{k-1} x^{a_i} \tag{3}$$

where a_i are nonnegative integers and $0 \leqslant a_0 < a_1 < \cdots < a_{k-1} \leqslant n-1$, we use $T_k(x)$ to denote the polynomial $1 + x + \cdots + x^{k-1}$ and use $f(x) \mid g(x)$ to represent $g(x) \equiv 0 \pmod{f(x)}$.

In [4], it is shown that a g-circulant A is a solution to (1) if and only if

$$Q(x)Q(x^g)\cdots Q(x^{g^{m-1}}) \equiv \lambda T_n(x) \qquad (\mathrm{mod}\ x^n - 1) \tag{4}$$

where $k^m = \lambda n$.

The following lemma is the starting point of this research.

Lemma 1 Let g be an integer, $Q(x)$ be of the form (3), $k^m = \lambda n$ and $c = (g, n)$, then the following four congruences are equivalent.

$$Q(x)Q(x^g)\cdots Q(x^{g^{m-1}}) \equiv \lambda T_n(x) \qquad (\mathrm{mod}\ x^n - 1) \tag{4}$$

$$Q(x)Q(x^g)\cdots Q(x^{g^{m-1}}) \equiv 0 \qquad (\mathrm{mod}\ T_n(x)) \tag{5}$$

$$Q(x)Q(x^c)\cdots Q(x^{c^{m-1}}) \equiv \lambda T_n(x) \qquad (\mathrm{mod}\ x^n - 1) \tag{6}$$

$$Q(x)Q(x^c)\cdots Q(x^{c^{m-1}}) \equiv 0 \qquad (\mathrm{mod}\ T_n(x)) \tag{7}$$

It is evident that (4) is equivalent to (5), and (6) is equivalent to (7). From Corollary 1.2 in [7], we can get the equivalence between (4) and (6).

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Therefore, A is a solution to (1) if and only if one of the four congruences holds.

Now, we state without proof a proposition which covers the main results in [4,5,6].

Proposition 2 Let $Q(x)$ be the Hall Polynomial of a g-circulant A.

i) Suppose $g^m \equiv 0 \pmod{n}$, then A satisfies (1) iff $T_c(x) \mid Q(x)$, where $c = (g, n)$.

ii) Suppose $Q(x) \equiv T_r(x) \pmod{x^n - 1}$, then A satisfies (1) iff $rg^{m-1} \equiv 0 \pmod{n}$ and $r \equiv 0 \pmod{c}$, where $c = (g, n)$.

Now, we can construct a new class of solutions to (1) via Proposition 2.

Let c be a factor of n, and there be such a positive integer s that $c^s \equiv 0 \pmod{n}$. If $s \leqslant m$, the g-circulants A satisfying (1) are characterized by (i) of Proposition 2. If $s > m$, we have an integer sequence $\{K_0, K_1, \cdots, K_{t-1}\}$ such that $K_0 = 0$, $0 < K_{i+1} - K_i \leqslant m$, $i = 0, 1, \cdots, t-2$, and $K_{t-1} \geqslant s - m$.

Put

$$Q_0(x) \equiv T_c(x) T_c(x^{c^{K_1}}) \cdots T_c(x^{c^{K_{t-1}}}) \qquad \pmod{x^n - 1}. \tag{8}$$

If we remove the restriction that A is a $(0,1)$ g-circulant in following Theorem 3, we can give more general solutions of (1), of course, $(0,1)$ g-circulants as the special case of the solutions are involved in the following Theorem 3.

Theorem 3

$$Q_0(x) Q_0(x^c) \cdots Q_0(x^{c^{m-1}}) \equiv 0 \qquad (\mathrm{mod}\ T_n(x)). \tag{9}$$

Proof A simple calculation shows that

$$Q_0(x) Q_0(x^c) \cdots Q_0(x^{c^{m-1}}) = \prod_{i=0}^{m-1} \Big(\prod_{j=0}^{t-1} T_c(x^{c^{K_j + i}}) \Big)$$

$$= \prod_{j=0}^{t-1} \Big(\prod_{i=0}^{m-1} T_c(x^{c^{K_j + i}}) \Big) = \prod_{j=0}^{t-1} T_{c^m}(x^{c^{K_j}})$$

$$= \frac{x^{c^m} - 1}{x - 1} \, \frac{x^{c^{K_1 + m}} - 1}{x^{c^{K_1}} + 1} \cdots \frac{x^{c^{K_{t-1} + m}} - 1}{x^{c^{K_{t-1}}} - 1}$$

$$= T_{c^{K_{t-1} + m}}(x) \prod_{j=0}^{t-2} \frac{x^{c^{K_j + m}} - 1}{x^{c^{K_{j+1}}} - 1}$$

Since $K_{j+1} \leqslant K_j + m$, $x^{c^{K_{j+1}}} - 1 \mid x^{c^{K_j + m}} - 1$, it follows that $h(x) = \prod_{j=0}^{t-2} \frac{x^{c^{K_j + m}} - 1}{x^{c^{K_{j+1}}} - 1}$ is a polynomial with integer coefficients such that

$$Q(x) Q(x^c) \cdots Q(x^{c^{m-1}}) \equiv T_{c^{K_{t-1} + m}}(x) h(x)$$

Since $s \leqslant k_{t-1} + m$, $c^s \mid c^{K_{t-1} + m}$ implies that $n \mid c^{K_{t-1} + m}$, thus, $T_n(x) \mid T_{c^{K_{t-1} + m}}(x)$ We get (9), and the proof is complete.

Now, let $(s,n)=1$, put

$$Q(x)=T_d(x)T_d(x^{c^{K_1}})\cdots T_d(x^{c^{K_{t-1}}}) \pmod{x^n-1}, \tag{11}$$

where $d=sc$. In the same way like Theorem 3, we have

Theorem 4

$$Q_1(x)Q_1(x^c)\cdots Q_1(x^{c^{m-1}})\equiv 0 \pmod{x^n-1} \tag{11}$$

When $t=1$, we get (ii) of Proposition 2.

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关于矩阵方程$A^m=\lambda J$的一些结果

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摘　要

本文利用四个等价的同余式得到命题2,从而概括了别人的一些结果。

命题 2　设$Q(x)$是g 循环矩阵A的Hall多项式

（1）假设 $g^m\equiv 0 \pmod n$，则A满足$A^m=\lambda J$ 当且仅当$T_c(x)\mid Q(x)$，$c=(g,n)$；

（2）假设 $Q(x)=T_r(x) \pmod{x^n-1}$，则A满足$A^m=\lambda J$当且仅当$rg^{m-1}=0 \pmod n$和$r\equiv 0 \pmod c$.它们的Hall-多项式如下：

在此基础上得到二组新解

$$Q_0(x)\equiv T_c(x)T_c(x^{c^{K_1}})\cdots T_c(x^{c^{K_{t-1}}}) \pmod{x^n-1}$$

$$Q_1(x)\equiv T_d(x)T_d(x^{c^{K_1}})\cdots T_d(x^{c^{K_{t-1}}}) \pmod{x^n-1}$$